MA403 Question and Answers

tutorial-1 :

1.1 1. $x \neq 0$
$y.1 = y(xx^{-1})$
$= (xy)x^{-1}$
$= (xz)x^{-1}$
$= z$

2. $y + 0 = y + x + x^{-1}$
$= x + y + x^{-1}$
$= x + z + x^{-1}$
$= z$

3. $0x + 0x = (0+0)x$
$= 0x$
$\Rightarrow 0x + 0x = 0x + 0$
$\Rightarrow 0x = 0$ (from 2)

4. $(-x) + (-)(-x) = 0 = x + (-x)$
$(-)(-x) = x$ (2)

5. $y < z$ $x > 0$
then $(z-y) > 0$
$x.(z-y) > 0$
$\Rightarrow xz - xy > 0$
$\Rightarrow xz > xy$

6. $x \neq 0$ as $xy = xz$ then $y = z$
$(x)(\frac{1}{x}) = 1 = (\frac{1}{x})(\frac{1}{\frac{1}{x}})$
$(x)(\frac{1}{x}) = (\frac{1}{\frac{1}{x}})(\frac{1}{x})$
$\Rightarrow x = \frac{1}{(\frac{1}{x})}$

7. $(-x)(-y)$
finding
$(-x)(y) + (x)(y)$
$= (-x + x)(y)$
$= 0.y = 0$
so $-(xy) = (-x)(y)$
now $(-x)(-y) = -(x(-y))$
$= -(-y.x)$
$= -(-(xy))$
$= xy$

8. If $x \neq 0$ $x > 0$ or $x < 0$
for $x > 0$
$\Rightarrow x.x > 0$

$x + (-x) = 0 < x \Rightarrow -x < 0$
$-x > 0$
$\Rightarrow (-x)(-x) > 0$
$\Rightarrow x^2 > 0$

so $1 > 0$

1.2 let $r \in \mathbb{Q}$ then $r = \frac{m}{n}$, $\gcd(m,n) = 1$
now $r^2 = 5 = \frac{m^2}{n^2}$
$5n^2 = m^2$
or $5 | m^2$
$\Rightarrow 5 | m$
$m = 5k$
sim $n = 5f$
so $\gcd(m,n) \neq 1$ $\ast$

$5 | m^2 \Rightarrow 5 | m$
proof:
$5 | m^2$ and $5 \nmid m$
then
$m = 5k + r$, $r = 1,2,3,4$
as but $5 \nmid m^2$ $\ast$

1.3 $x, y \in \mathbb{R}$, $x < y$
now $y - x > 0$
from archimedian prop.
$\exists n \in \mathbb{N}$ s.t
$n(y-x) > 1$
$\Rightarrow ny > nx + 1$

$\exists m_1 \in \mathbb{N}$ $m_1 \geq nx$
$\exists m_2 \in \mathbb{N}$ $m_2 \geq nx$
so $-m_2 \leq nx \leq m_1$
$m-1 \leq nx < m$
$nx < nx + 1 < m' < ny$
Break into 1 length

now for $S \subseteq \mathbb{Z}$ s.t $S = \{m \in \mathbb{Z} \mid m \leq M, M \geq nx + 1\}$
then it has a sup $= nx + 1$
then $nx < m < nx + 1$

so $nx < m < ny$ $\qquad x < \frac{m}{n} < y$

$\Rightarrow x < \frac{m}{n} < y$

1.4 $0 \leq x < \frac{1}{n} \quad \forall n \in \mathbb{N}$

let's suppose $x > 0$, now by archemedicity property $\exists n_0 \in \mathbb{N}$

s.t

$n_0 x > 1$

$\Rightarrow x > \frac{1}{n_0}$ $\quad *$

so $x > 0$

$\Rightarrow x \leq 0$

as $x \geq 0$ and $x \leq 0$

$\Rightarrow x = 0$

1.5 $\mathbb{R}$ is uncoutable :

let $f: \mathbb{N} \to (0,1)$ (Bijection)

s.t

$x_1 = 0.x_{11}x_{12}x_{13}\ldots$

$x_2 = 0.x_{21}x_{22}x_{23}\ldots$

where $x_{ij} = 0$ or 1

and $\bar{x}_{ij} = \begin{cases} 0; & x_{ij} = 1 \\ 1; & x_{ij} = 0 \end{cases}$

now $\exists\ 0.\bar{x}_{11}\bar{x}_{22}\ldots$

for which as $\bar{x}_{ii} \neq x_{ii}$

$0.\bar{x}_{11}\bar{x}_{22}\ldots \neq x_i \quad \forall i \in \mathbb{N}$

$*$ $\quad \therefore f: \mathbb{N} \to (0,1)$ is not surjective

$\therefore f: \mathbb{N} \to (0,1)$ is not bijection

$\therefore |\mathbb{N}| < |\mathbb{R}|$

Tutorial-2 :

3.2 (a) All int: $\{n \mid n \in \mathbb{N}\}$ in $\mathbb{R}$
here, no limit/acc. points.
not open as all points are not interior points.

$(\mathbb{N})^c = (-\infty, 1) \cup (1,2) \cup (2,3) \cup \ldots\ldots$

union of countable open sets
$\Rightarrow$ open
$\therefore \mathbb{N}^c$ is open
$\Rightarrow$ $\mathbb{N}$ is closed

(b) $(a,b]$, all accumulation points $= [a,b]$
all interior points $= (a,b)$
as $(a,b) \neq (a,b]$
not open

also $(a,b]^c = (-\infty, a] \cup (b, \infty)$
as not open
$(a,b]$ not closed

(f) $(-1)^n + \frac{1}{m}$, $(m,n = 1,2,\ldots)$
limit points $= \{-1+0, +1+0\}$
$= \{-1, 1\}$

$\left\{-1+\frac{1}{m} \,\middle|\, m=1,2,\ldots\right\} \cup \left\{1+\frac{1}{m} \,\middle|\, m=1,2,\ldots\right\}$, not

$S \cup S' = \bar{S} = \left\{-1+\frac{1}{m}\right\} \cup \left\{+1+\frac{1}{m}\right\} \cup \{0\} \subsetneq S$
$\therefore$ not closed

not open as $-1+\frac{1}{m}$ not interior point but in S.

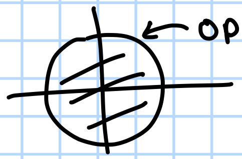
(g) $\left(\frac{1}{m}\right) + \left(\frac{1}{n}\right)$

$\bar{S} = \left\{\frac{1}{m}+\frac{1}{n}\right\} \cup \underbrace{\left\{\frac{1}{m}\right\} \cup \left\{\frac{1}{n}\right\} \cup \{0\}}_{\text{limit points}} \subsetneq S$

$\therefore$ not closed
also not open since $\frac{1}{m}+\frac{1}{n}$ is not an interior point
(open if all points = interior p)

3.3

(d) $x^2+y^2<1$



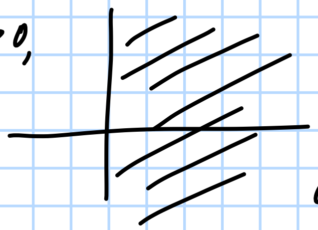
$B(0,1)$

open

limit points: $x^2+y^2 \leq 1$ $\quad \bar{S} = \{(x,y) \mid x^2+y^2 \leq 1\}$

not closed as $\bar{S} \subsetneq S$

(e) $x>0$,



$S = \{(x,y) \mid x>0\}$

$\bar{S} = \{(x,y) \mid x \geq 0\}$

as $\bar{S} \subsetneq S$ not-closed

open

(f) (x,y), $x \geq 0$

$\bar{S} = \{(x,y) \mid x \geq 0\} \subseteq S$

$\therefore$ closed

not open $(0,0) \in S$ but not an int. point

3.5 R^1 $\quad T = R^1 - S \quad S$

both S and T are open and closed

if $S \neq R^1$ and $S \neq \phi$

let $\dfrac{s_0+t_0}{2}$ in S or T

since $R = S \cup T$

$\dfrac{s_0+t_0}{2} \in S$ then $\dfrac{s_0+t_0}{2} = s_1$

or $= t_1$

$\{s_n\} \subseteq S \quad \{t_m\} \subseteq T$

however it will converge to same point

as $S \cap T = \phi$ ⨳

Note: true for all $\mathbb{R}^n$

3.12 S' derived set

↳ set of all limit points

$\bar{S} = S \cup S'$

(a) a set is closed $\Leftrightarrow$ all limit points in set

as $S' =$ set of all limit points $\Rightarrow$ it is closed

(b) $S \subseteq T$ then $S' \subseteq T'$

$\forall x \in S'$ x is a limit point of S.

if x is a limit point of S then

$\forall r > 0$, $B(x, r)$ has a point in S other than x.

now if $S \subseteq T$ then

$B(x, r)$ has point in T other than x, so x is a limit point for T.

$\therefore$ $x \in S' \Rightarrow x \in T'$

$\Rightarrow$ $S' \subseteq T'$

3.13 ① $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$

<u>proof</u> : $x \in \overline{S \cap T}$

$\Rightarrow x \in S \cap T$ or $(S \cap T)'$

$\Rightarrow x \in S$ and T or S' and T'

$\Rightarrow x \in S$ or S', and $x \in T$ or T'

$\Rightarrow x \in S \cup S'$ and $T \cup T'$

$\Rightarrow$ $x \in \overline{S} \cap \overline{T}$

3.12 > $S \subseteq T \Rightarrow S' \subseteq T'$ so $\overline{S} \subseteq \overline{T}$

now using this

(a) $S \cap T \subseteq T \Rightarrow \overline{S \cap T} \subseteq \overline{T}$

$S \cap T \subseteq S \Rightarrow \overline{S \cap T} \subseteq \overline{S}$

$\Rightarrow \overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$

② $S \cap cl(T) \subseteq cl(S \cap T)$ if S is open

<u>proof</u>: if S is open, then $\forall x \in S$ $\exists d > 0$ s.t

$B(x, d) \subseteq S$

$B(x, r) \cap S \subseteq B(x, r)$, $r \leq d$

and $B(x, r) \cap S \subseteq B(x, d)$, for $r > d$

now, if $x \in S \cap cl(T)$

$\Rightarrow x \in cl(T)$

$\Rightarrow$ $B(x, k) \cap T \neq \phi$ for any $k > 0$.

for $r \leq d$

now $B(x, r) \cap (S \cap T) = (B(x, r) \cap S) \cap T = B(x, r) \cap T \neq \phi$ if $r \leq d$

$= B(x, d) \cap T \neq \phi$ if $r > d$

$\therefore$ $x \in cl(S \cap T)$

3.15 $\mathbb{R}^n$ $F = \{A \mid A \in \mathbb{R}^n\}$

$S = \bigcup_{A \in F} A$ $T = \bigcap_{A \in F} A$

(a) x is acc point of T then x is acc point of each set A in F

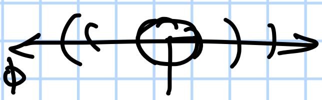
<u>True</u>

as

$(B(x, r) - \{x\}) \cap T \neq \phi$

$T \subseteq A$

$(B(x, r) - \{x\}) \cap A \neq \phi$



rmin const

(b) x is acc point S

$(B(x, r) - \{x\}) \cap S \neq \phi$

<u>No</u>: set consisting single point $x \in \mathbb{R}^n$, $S = \bigcup_{A \in F} A$

x is acc point but not in set A in F.

3.16 S of $\mathbb{Q}$ in $(0,1)$ $\neq \bigcup S'$ ← open set
↳ countable collection

$S = \{x_1, x_2, \ldots\}$

$S = \bigcap_1^\infty S_k$ each S_k is open

$\{Q_n\}$ s.t $Q_{n+1} \subseteq Q_n \subseteq S_n$ s.t $x_n \notin Q_n$

$x_1 \in I_1 \subseteq S_1$ $Q_1 \subseteq I_1$ ← Bounded + closed

s.t $x_1 \notin Q_1$

let x_2 in θ_1 $x_2 \in S_2$

$Q_1 \to$ inf rationals

call x_2

$x_2 \in S_2$

$I_2 \subseteq S_2$ and $I_2 \subseteq Q_1$

$Q_2 \subseteq I_2$

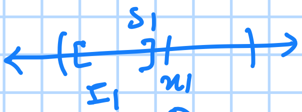
$x_2 \notin Q_2$

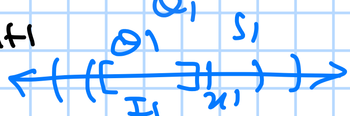
Q_n is closed

$Q_n \subseteq Q_{n-1} \subseteq S_{n-1}$

$x_n \notin Q_n$

$x_{n+1} \notin Q_{n+1}$





$\bigcap_{n=1}^{n=\infty} Q_n \neq \phi$

as $S \cap (\bigcap_{n=1}^{n=\infty} Q_n) = \phi$

as $x_n \notin Q_n$

suppose $\mathbb{Q} \cap (0,1) = \bigcap_{n=1}^\infty S_n$ where S_n's are open.

let $\mathbb{Q} \cap (0,1)$ be enumerated as $\{x_1, x_2, \ldots\}$

since S_1 is open, $\exists$ an open interval θ_1 s.t $x_1 \in \theta_1$. so, $\exists$ a non-trivial closed interval $I_1 \subseteq \theta_1$ and $x_1 \notin I_1$

open ball (interior point)

since there are elements of $\mathbb{Q} \cap (0,1)$ in I_1

$S_2 \cap I_1 \neq \phi$

so, $\exists$ an non-empty open interval

$\theta_2 \subseteq S_2 \cap I_1$

Thus $\exists$ a non-trivial closed interval

$I_2 \subseteq \theta_2 \subseteq I_1$

and $x_2 \notin I_2$

we get $I_1, I_2, I_3, \ldots$

s.t $I_n \subseteq I_{n-1}$
- Non empty
- closed
- $x_n \notin I_n, \forall n \in \mathbb{N}$

By cantor's thm

$\bigcap_{n=1}^\infty I_n \neq \phi$, also $\bigcap S_n \supseteq \bigcap_{n=1}^\infty I_n$

$\bigcap_{n=1}^\infty I_n$ avoids all elements of S

ALSO, $(\bigcap_{n=1}^\infty I_n) \cap S = \phi$ ⚡

3.17 $S \subseteq \mathbb{R}^n$ coll of isolated points of S is cont.

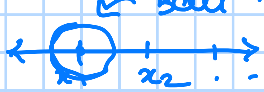
isolated points $= F$, $x \in F$ then

$(B(x, r_x) - \{x\}) \cap S \neq \phi$

now if $\mathbb{Q}^n = \{x_1, x_2, \ldots\}$

now $(B(x, r_x) - \{x\}) \cap S \neq \phi$

let $\mathbb{Q} = \{A_1, A_2, \ldots\}$ all balls of rational radii and rational centre point



Uall such balls = covering

$\therefore \exists$ a countable subcovering

$\Rightarrow x_i$ are countable

now, as $(B(x, r_x) - \{x\}) \cap S \neq \phi$

$\bigcup_{x \in \mathbb{Q}^n} (B(x, r_x) - \{x\})$ is a covering of S.

From Lindelöf covering theorem

countable covering, $\therefore$ $x \in \mathbb{Q}^n$ countable

$\therefore$ Isolated points are countable.

3.20 $S = \mathbb{R}^1$

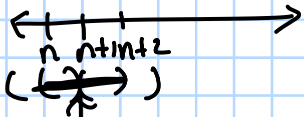
then S is closed,
not bounded

$I = \{ I_n \mid I_n = (n, n+2), \forall n \in \mathbb{Z} \}$

$\bigcup_{A \in I} A \supseteq \mathbb{R}$

$\therefore$ I is open covering

s.t if we don't have one $(n, n+2)$



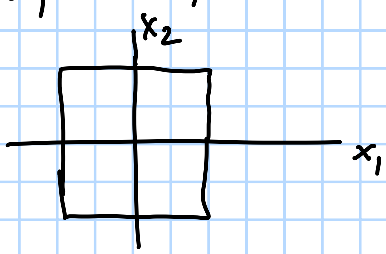
$\{n+1\}$ not in $\bigcup_{A \in F} A$

$\therefore$ we cannot remove

3.27 (a) $d_1(x,y) = \max\{|x_1-y_1|, |x_1-y_2|, |x_2-y_1|, |x_2-y_2|\}$

$B(a,r) = \{x \mid d(a,x) < r\}$

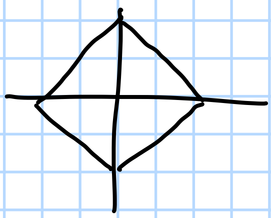
$B(\bar{0},1) = \{x \mid d(\bar{0}, x) = \max\{|x_1|, |x_2|\} < 1\}$



(b) $d_2(x,y) = \sum |x_i - y_i|$

$B(\bar{0},1) = \{x \mid d_2(\bar{0}, x) < 1\}$

$= \{x \mid |x_1| + |x_2| < 1\}$



(c) $|x_1| < 1$ $|x_2| < 1$ $|x_3| < 1$

(d) $|x_1| + |x_2| + |x_3| < 1$

3.29 $d'(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$

① $d'(x,x) = 0$
as $d(x,x) = 0$

② $d'(x,y) = d'(y,x)$

③ $d'(x,y) > 0$ for $x \neq y$
as $d(x,y) > 0$

④ $d'(x,y) + d'(y,z) \geq d'(x,z)$

$$\underline{\frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}} \quad \underline{\frac{d(x,z)}{1+d(x,z)}}$$

Note : $d(x,y)+d(y,z) \geq d(x,z)$

$\Rightarrow$ $1+d(x,y)+d(y,z) \geq 1+d(x,z)$

$\Rightarrow$ $-\dfrac{1}{1+d(x,y)+d(y,z)} \geq \dfrac{-1}{1+d(x,z)}$

$\Rightarrow$ $1-\dfrac{1}{1+d(x,y)+d(y,z)} \geq 1-\dfrac{1}{1+d(x,z)}$

$\Rightarrow$ $\dfrac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \geq \dfrac{d(x,z)}{1+d(x,z)}$

$\Rightarrow$ $\dfrac{d(x,y)}{1+d(x,y)+d(y,z)} + \dfrac{d(y,z)}{1+d(x,y)+d(y,z)} \geq \dfrac{d(x,z)}{1+d(x,z)}$

$\Rightarrow$ $d'(x,y) + d'(y,z) \geq d'(x,z)$

3.30 To prove: Every finite subset of metric space is closed

<u>proof</u>:

Let $S \subseteq M(\mathbb{R}^n, d)$ s.t
S is finite then

$S = \{x_1, x_2, \ldots x_n\}$

now let <u>S be open</u>.

$\exists\, x \in F \setminus S$ s.t x is a limit point for S.

$\forall\, r>0$, $(B(x,r) - \{x\}) \cap S \neq \phi$

But if
$r < \min\{|x_i - x_j|\}$
then this is not possible

$\therefore$ x is not a limit point.

$\therefore$ no limit point outside

$\therefore$ S is closed

3.31 $\bar{B}(a,r) = \{x ; d(x,a) \le r\}$

(a) <u>To prove</u>: $\bar{B}(a,r)$ is closed

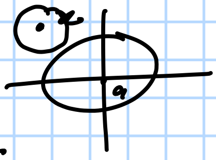
<u>proof</u>:

let $x \in M - \bar{B}(a,r)$

then

$d(x,a) > r$

let $\delta = \frac{d(x,a)-r}{2}$



now $\forall y$ in $B(x,\delta)$

$$d(y,a) \ge d(y,x) + d(x,a) \ge d(x,y) + r + \delta$$

$$d(y,a) \ge d(x,y) + \cancel{r} + \frac{d(x,a)+r}{2}$$

$$= d(x,y) + \frac{d(x,a)}{2} + \frac{r}{2} > \frac{d(x,a)}{2} + \frac{r}{2}$$

$$> \frac{r}{2} + \frac{r}{2} = r$$

$\therefore$ $y \in M - \bar{B}(a,r)$

$\therefore$ $B(x,\delta) \subseteq M - \bar{B}(a,r)$

$\therefore$ $\exists\, \delta' > 0$ s.t. $\forall x \in M - \bar{B}(a,r)$

$B(x,\delta') \subseteq M - \bar{B}(a,r)$

$\therefore$ $M - \bar{B}(a,r)$ is open

$\therefore$ $\bar{B}(a,r)$ is closed

(b) $B(a,1) = \{a\}$ for $d(x,y) = \begin{cases} 1; & x \ne y \\ 0; & x = y \end{cases}$

$$\bar{B}(a,1) = M$$

3.32 $A \subseteq S \subseteq \bar{A}$

$S \subseteq T \subseteq \bar{S}$

then

$A \subseteq T \subseteq \bar{A}$

<u>proof</u>: let $y \in \bar{S}$, then either $y \in S$ or $y \in S'$

<u>Case I</u>: $y \in S \Rightarrow y \in \bar{A}$

$\bar{S} \subseteq \bar{A}$

<u>Case II</u>: $y \in S'$ then

$\forall r > 0$

$(B(y,r) - \{y\}) \cap S \ne \phi$

as $S \subseteq \bar{A}$

$(B(y,r) - \{y\}) \cap \bar{A} \ne \phi$

$\therefore$ y is a limit point for $\bar{A}$

$\therefore \quad y \in \overline{A}$

$\overline{S} \subseteq \overline{A}$

now, $A \subseteq T \subseteq S \subseteq \overline{S} \subseteq \overline{A}$

$\Rightarrow \quad A \subseteq T \subseteq \overline{A}$

Tutorial-3:

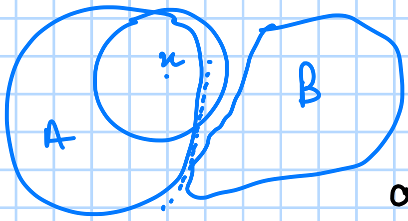
1. (a) A, B disjoint open sets in X.

$A \cap B = \phi$
and open
so $\exists r > 0$ s.t $x \in A$
$B(x, r) \subseteq A$

$\exists r' > 0$ s.t $y \in B$
$B(y, r') \subseteq B$

seperated if $\bar{A} \cap B = \phi$ and
$A \cap \bar{B} = \phi$

$A' =$ set of limit points in A
let $x \in A'$ be a limit point in A
then
$\forall r > 0$ $(B(x, r) - \{x\}) \cap A \neq \phi$



x is s.t
x is not in A but
in A', then
as $\forall r > 0$, $(B(x, r) - \{x\})$
will have inf many
points in A and as
$A \cap B = \phi$
this mean that
$\forall r > 0$, $\exists r_0$ s.t

$B(x, r_0) \subseteq B$
as there are
points in $B(x, r_0)$ s.t
they are in B.

$\therefore \bar{A} \cap B = \phi$
simullary $A \cap \bar{B} = \phi$

or $A \cap B = \phi$
and B^c is closed
$A \subseteq B^c$
also $\bar{A} \subseteq B^c$
as if $x \in \bar{A}$
then $\forall r > 0$, $(B(x, r) - \{x\}) \cap A \neq \phi$
so $\exists$ a point $y \in B(x, r) - \{x\}$
s.t $y \in A$
as $y \in A \Rightarrow y \in B^c$
$\Rightarrow \bar{A} \subseteq B^c$

$\therefore \bar{A} \subseteq B^c$
and as $B \cap B^c = \phi$
$B \cap \bar{A} = \phi$
also $A \cap \bar{B} = \phi$

(b) $d(x,\cdot)$ is a cont function

as for some y let :

let $f(x) = d(x,y)$

then for $x = c$

$\lim_{h \to 0} f(c+h) = d(c+h, y)$

Note

$d(c-h,y) \leq d(x,y) \leq d(c+h,y)$

or vice versa

for $h \to 0$

$d(x,y) = d(c,y)$

$\therefore$ f is cont at c.

now

$$C = \left\{ x \,\middle|\, f(x) < x \right\}$$

if $f(x)$ is a cont function

then by definition of $f(x)$

for $x \in C$ $\quad$ $f(x) < x$ and $f(x)$ is cont

$\exists\ \delta > 0$ s.t

$B(x, \delta) \subseteq C$

as $f(x) < x$

$$B(x,\delta) = \left\{ y \,\middle|\, d(x,y) < \delta \right\}$$

$f(x-\delta) < x$

and

$f(x+\delta) < x$

as $\|x - y\| < \delta$

$y \in (x-\delta, x+\delta)$

$y \in C$

$\therefore$ $B(x,\delta) \subseteq C$

$\therefore$ C is open.

$\therefore$ if $f(x)$ is cont

$C = \{x \mid f(x) < x\}$ is open.

now $A = \left\{ g \in X \,\middle|\, d(P,g) < \delta \right\}$

let $x \in A$ then

$d(P,x) < \delta$

also $\quad$ $\exists\ r > 0$ s.t

$$B(x, r) = \left\{ y \,\middle|\, d(x,y) < r \right\}$$

for $r = \delta$

as $B(x,\delta) = \{ y \mid d(x,y) < \delta \}$

as $\quad$ $d(x,P) \geq d(x,y) + d(y,P)$

$d(x,P) - d(x,y) \geq d(y,P)$

$d(P,y) \leq d(x,P) - d(x,y)$

$d(P,y) < \delta - d(x,y)$

as $d(x,y) < \delta$

$d(P,y) < \delta - d(x,y) < \delta$

$d(x,y) > 0$

$\therefore$ $y \in A$

$\therefore \quad B(x,y) \subseteq A$
$\therefore \quad A$ is open.

now, $B = \{ g \in X \mid d(g,p) > \delta \}$

$x \in B$ then $d(x,p) > \delta$
we take
$0 < r \leq d(x,p) - \delta$
then
$B(x,r) = \{ y \mid d(x,y) < r \}$

$d(x,y) < r$
$d(x,p) - d(y,p) \leq d(x,y) < r$
$d(x,p) < r + d(y,p)$
$\cancel{d(x,p)} < \cancel{d(x,p)} - \delta + d(y,p)$
$\delta < d(y,p)$
$\therefore y \in B$
$\therefore B(x,r) \subseteq B$
$\therefore B$ is open

as $A \cap B = \phi$ and A and B are open
A, B are seperate.

(c) Here let $x_1, x_2 \in X$
let
$r = d(x_1, x_2) = d(x_2, x_1)$
let $t \in (0,1)$
now,
$A_t = \{ y \mid d(x_1, y) < tr \}$
$B_t = \{ y \mid d(x_1, y) > tr \}$

Note: $A_t \cap B_t = \phi$ and
A_t, B_t are open
Note A_t and B_t are non empty as $x_1 \in A_t$ and $x_2 \in B_t$.
So, A_t and B_t are seperate.

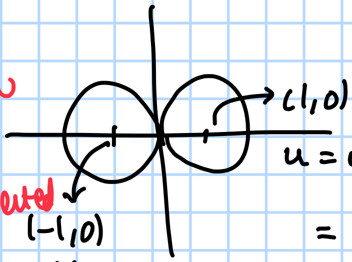
Now $A_t \cup B_t = X$, but as X is connected

$X \neq A_t \cup B_t$
$\Rightarrow A_t \cup B_t \subseteq X$
so there is a point in X which is not in A_t, B_t.

That point is s.t $d(x,p) = tr$

as t is uncoutable, there are uncoutable p.

2.
see how to prove if sets are connected



u = open ball at $(1,0)$ of radius $=1$
$= B((1,0),1)$

v = closed ball at $(-1,0)$ of radius 1
$= \bar{B}((-1,0),1)$

as $\bar{u} \cap v = \{(0,0)\}$, u and v are not seperate.
let $X = u \cup v$ (this is connected)

But Balls of $1/2$ with same centre are not.

3.40 Let A_i be a compact subsets.

$$\bigcap_{i=1}^{\infty} A_i \subseteq A_j \text{ for any } j$$

as A_j is compact, $\bigcap_{i=1}^{\infty} A_i \subseteq A_j$ = covering F

then covering F also covers $\bigcap_{i=1}^{\infty} A_i$

3.41 Let A_K for $K = 1, 2, \dots n$ be compact sets.

Let F is the open covering of $\bigcup_{i=1}^{n} A_i$.

If there doesn't exist a finite subcovering of $\bigcup_{i=1}^{n} A_i$, then

$\exists\, i \in \mathbb{N}$ s.t A_i doesn't have a finite subcovering.

But as A_i is compact. $\#$

$\therefore \bigcup_{i=1}^{n} A_i$ has a finite subcovering.

3.42 $S = (a,b)$ is bounded in $\mathbb{Q}$. Now

Let $x \in \mathbb{Q} - S$, then

$x < a$ or $x > b$

if $x < a$ then $B(x,d) = (x-(a-x), x+(a-x))$

$= (2x-a, a)$

$2x < 2a$

$< a$

$\therefore (2x-a, a) \subseteq \mathbb{Q} - S$

similarly $B(x,d)$ for $d = x - d$

$(\cancel{x} - (\cancel{x} - d), x + (x-d))$

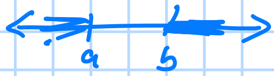
$= (d, 2x-d) \subseteq \mathbb{Q} - S$

$\therefore \mathbb{Q} - S$ is open

$\therefore S$ is closed.

$(a,b) \cap \mathbb{Q}$

then



$((a,b) \cap \mathbb{Q})^c$ is open

$\therefore (a,b) \cap \mathbb{Q}$ closed

$a < \quad < b$

$\therefore$ Bounded

b/w two rationals

$\downarrow$

irrational

so covering in has infinite subsets $\therefore$ not compact

Now to see that S is not compact, we know that $\mathbb{Q}$ is not complete.

So, to covers S we need infinite many coverings.

$\therefore S$ is not compact.

between two rationals, there is an irrational

Let a, b be rational $(a \neq b)$

then

WLOG $a < b$

$$\frac{m}{n} < \frac{p}{q}$$

$a < a + \frac{(b-a)}{2} < b$

$0 < \frac{b-a}{2} < \sqrt{2}\,\frac{(b-a)}{2} < b-a$

$0 < \sqrt{2}\,\frac{(b-a)}{2} < b-a$

$a < a + \sqrt{2}\,\frac{(b-a)}{2} < b$

$\therefore$ irrational b/w a, b

Tutorial-4-

If $a_n \to 0$ and $\{c_n\}_{n=1}^{\infty}$ is bounded, then $\{c_n a_n\}_{n=1}^{\infty} \to 0$

If $a_{n+2} = \frac{a_{n+1}+a_n}{2}$, $\forall n \geq 1$ show: $a_n \to \frac{a_1+2a_2}{3}$

$a_{n+2}-a_{n+1}$ $\quad a_{n+2}-a_{n+1} = \frac{a_n}{2} - \frac{a_{n+1}}{2}$

$$\cancel{a_3} - a_2 = \frac{a_1}{2} - \cancel{\frac{a_2}{2}}$$
$$\cancel{a_4} - \cancel{a_3} = \cancel{\frac{a_2}{2}} - \cancel{\frac{a_3}{2}}$$
$$\cancel{a_5} - \cancel{a_4} = \cancel{\frac{a_3}{2}} - \cancel{\frac{a_4}{2}}$$
$$\vdots$$
$$a_{n+2} - \cancel{a_{n+1}} = \cancel{\frac{a_n}{2}} - \frac{a_{n+1}}{2}$$
$$a_{n+2} - a_2 = \frac{a_1}{2} - \frac{a_{n+1}}{2}$$

as a_n converges
$$a_n - a_2 = \frac{a_1}{2} - \frac{a_n}{2}$$
$$\frac{3a_n}{\cancel{2}} = \frac{a_1}{\cancel{2}} + \frac{2a_2}{\cancel{2}}$$
$$a_n = \frac{a_1+2a_2}{3}$$

If converges as odd seq, even seq converges.
① bounded
② monotonic
} even and od

or $a_{n+2} - a_{n+1} = -\frac{1}{2}(a_{n+1}-a_n) = (-\frac{1}{2})(-\frac{1}{2})(a_n - a_{n-1}) \cdots\cdots = (-1)^n \frac{1}{2^n}(a_2-a_1)$

as $\frac{1}{2^n} \to 0$
$(-1)^n \frac{1}{2^n} \to 0$
$\frac{(-1)^n(a_2-a_1)}{2^n} \to 0$

$$\sum_{k=1}^{n} a_k - a_{k-1} = \sum_{k=1}^{n} \frac{(-1)^k(a_2-a_1)}{2^k}$$
$$\sum_{k=1}^{n} a_k - a_{k-1} = (a_2-a_1)\sum_{k=2}^{n} \frac{(-1)^k}{2^k}$$
$$a_n = (a_2-a_1)\sum_{k=2}^{n}\frac{(-1)^k}{2^k} + a_1$$
$$= (a_2-a_1)\left[\ \ 1\ \ \right]\left[\frac{1-r^n}{1-r}\right] + a_1$$
see
$$= (a_2-a_1)\left(\ 1\ \right)\left(\frac{1}{\frac{3}{2}}\right) + a_1$$
$$= (a_2-a_1)\frac{2}{3} + a_1$$
$$a_n = \frac{2a_2}{3} + \frac{a_1}{3}$$

$s_1 = \sqrt{2}$
$s_{n+1} = \sqrt{2+\sqrt{s_n}}$ → we use monotonic and bounded
we will show s_n is bounded using induction

as $s_1 < 2$
$s_k < 2$
then $s_{k+1} = \sqrt{2+\sqrt{s_n}} < \sqrt{2+\sqrt{2}} < \sqrt{2+2} = 2$
$\therefore \forall n \in \mathbb{N}$ $s_n < 2$ (By induction)

$$s_{n+1} - s_n = \frac{(\sqrt{2+\sqrt{s_n}}^2 - \sqrt{2+\sqrt{s_{n-1}}}^2)}{(\sqrt{2+\sqrt{s_n}} + \sqrt{2+\sqrt{s_{n-1}}})} = \frac{(\sqrt{s_n} - \sqrt{s_{n-1}})}{k} = \frac{s_n - s_{n-1}}{k'} \cdots\cdots = \frac{s_2 - s_1}{k'''}$$

where $k''' > 0$, now $s_{n+1} - s_n$ sign depends on $s_2 - s_1$.

$s_1 = \sqrt{2}$

$s_2 = \sqrt{2+\sqrt{2}}$

where $s_2 - s_1 = \dfrac{\cancel{2} + \sqrt{2} - \cancel{2}}{(k''')} > 0$

$\therefore s_{n+1} - s_n > 0$

$\Rightarrow s_{n+1} > s_n$

$\therefore$ monotonically inc

$\lim\limits_{n\to\infty} \sqrt[n]{p} = 1$ where $p > 0$

$p^{\frac{1}{n}} = 1 + h$

$p = (1+h)^n = 1 + nh + \cdots\cdots h^n$

claim $0 < h < \frac{p}{n}$

if not then $h \leq 0$ or $h \geq \frac{p}{n}$

1. $s_1 = \sqrt{2}$

$s_n = \sqrt{2 + \sqrt{s_{n-1}}}$

now by induction

$s_1 < 2$

$s_2 = \sqrt{2+\sqrt{s_1}} < \sqrt{2+2} = 2$

$\vdots$

$s_k < 2$

then $s_{k+1} < \sqrt{2+2} = 2$

$\therefore s_n < 2, \forall n \in \mathbb{N}$

now $s_{n+1} - s_n = \sqrt{2+\sqrt{s_n}} - \sqrt{2+\sqrt{s_{n-1}}}$

$= \dfrac{\sqrt{s_n} - \sqrt{s_{n-1}}}{(k)}$ $\quad k > 0$

$= \dfrac{s_n - s_{n-1}}{k''}$ $\quad k'' > 0$

$\vdots$

$= \dfrac{s_2 - s_1}{k'''}$ $\quad k''' > 0$

$= \dfrac{\sqrt{2+\sqrt{\sqrt{2}}} - \sqrt{2}}{k'''} > 0$

$\therefore s_{n+1} > s_n$

or mon inc.

$\therefore$ as mon inc and bounded above

conv.

$s = \sqrt{2+\sqrt{s}}$

$s^2 = 2 + \sqrt{s}$

2. (i) $\lim_{n\to\infty} \frac{1}{n^P} = 0 \quad (P>0)$

$\exists n_0 \in \mathbb{N}$ s.t

$n_0 > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{P}}$

$\Rightarrow \left|\frac{1}{n_0^P}\right| < \varepsilon$

$\Rightarrow \left|\frac{1}{n^P} - 0\right| < \varepsilon \quad \forall n \geq n_0$

$\therefore \lim_{n\to\infty} \frac{1}{n^P} = 0$

(ii) $\lim_{n\to\infty} \sqrt[n]{P} = 1$

for $P>1$

$x_n = \sqrt[n]{P} - 1 > 0$

now, $1 + x_n = \sqrt[n]{P}$

$(1+x_n)^n = P$

also $(1+x_n)^n = 1 + nx_n + \cdots \geq 1 + nx_n$

$1 \leq 1 + n x_n \leq (1+x_n)^n$

$\Rightarrow 1 \leq 1 + nx_n \leq P$

$\Rightarrow x_n \leq \frac{P-1}{n}$

$\Rightarrow 0 \leq x_n \leq \frac{P-1}{n}$

by sandwich theorm

$\lim_{n\to\infty} x_n = 0$

$\lim_{n\to\infty} (\sqrt[n]{P} - 1) = 0$

$\Rightarrow \lim_{n\to\infty} \sqrt[n]{P} = 1$

if $P=1$, then trivial.

if $P<1$ then

$x_n = \frac{1}{\sqrt[n]{P}} - 1$ as $\frac{1}{P} > 1$

$\Rightarrow \frac{1}{(P)^{1/n}} > 1$

now similarly

$x_n = \left(\frac{1}{P}\right)^{\frac{1}{n}} - 1$

as $\frac{1}{P} > 1$ $x_n > 0$

and

we get

$\lim_{n\to\infty} \left(\frac{1}{P}\right)^{\frac{1}{n}} - 1 = 0$

$\Rightarrow \lim_{n\to\infty} (P)^{\frac{1}{n}} = 1$

3. $|a| < 1$

then $\lim_{n\to\infty} 1+a+\cdots a^n = \lim_{n\to\infty} \frac{a^n-1}{a-1} \to \frac{1}{1-a}$ → to show this

now from $\frac{n^\alpha}{(1+p)^n}$ $p>0$ tends to 0

as $n\to\infty$

as $(1+p)^n > \binom{n}{k} p^k$ (Binomial theorm)

then $(1+p)^n > \frac{(n)(n-1)\cdots(n-k+1)p^k}{k!}$

s.t $(n-k+1) > \frac{n}{2}$

$2n-2k+2 > n$

$n > 2k+2$

$k+1 < \frac{n}{2}$

$k < \frac{n}{2} - 1$

$(1+p)^n > \left(\frac{n}{2}\right)^k \frac{p^k}{k!}$

$0 \le \frac{1}{(1+p)^n} \le \frac{k!\,(2)^k}{(n)^k (p)^k}$

$0 \le \frac{n^\alpha}{(1+p)^n} \le \frac{2^k k!}{(p)^k\,(n)^{k-\alpha}}$

if $k-\alpha > 0$ then done $\underline{k > \alpha}$

as $n\to\infty$

$0 \le \frac{n^\alpha}{(1+p)^n} \le 0$

$\Rightarrow \lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

Now note: if $\alpha = 0$

then $\lim_{n\to\infty} \frac{n^0}{(1+p)^n} = \frac{1}{(1+p)^n} = 0$

also as $\left(\frac{1}{1+p}\right) < 1$

$\Rightarrow \left|\frac{1}{1+p}\right| < 1$

$\Rightarrow |a| < 1$

then $(a)^n = 0$

for $n\to\infty$

$\therefore \lim_{n\to\infty} \frac{a^n-1}{a-1} = \frac{-1}{a-1} = \frac{1}{1-a}$

4.1 : (a) $|z|<1$ then
for any $a \in \mathbb{R}$
$|a|<1$
then $(a)^n \to 0$
(Already proved)
then $|z|^n < \varepsilon$
for any z s.t
$|z|<1$
$\Rightarrow |z^n - 0| < \varepsilon$
$\therefore z^n \longrightarrow 0$

also as $|z^n - 0| < \varepsilon$
$\exists n_0 \in \mathbb{N}$ s.t
$n_0 \log\left(\frac{1}{|z|}\right) > \varepsilon'$
as $|z|<1 \Rightarrow \frac{1}{|z|} > 1 \Rightarrow \log\left(\frac{1}{|z|}\right) > 0$

$\Rightarrow n_0 > \dfrac{\varepsilon'}{\log\left(\frac{1}{|z|}\right)}$
$\Rightarrow n_0 \log\left(\frac{1}{|z|}\right) > \varepsilon'$
$\Rightarrow n_0 \log(|z|) < -\varepsilon'$
$\Rightarrow \log(|z|^{n_0}) < -\varepsilon'$
$\Rightarrow |z^{n_0} - 0| < e^{-\varepsilon'}$

or
$\left(N \geqslant \log_{|z|} \varepsilon - 1 \right)$
$\Rightarrow N+1 \geqslant \log_{|z|} \varepsilon$
$\Rightarrow |z|^{N+1} \leqslant \varepsilon$
as $|z|<1$

$|z|>1$ then
$|z|^2 > |z| > 1$
now as this is monotonically inc seq
$\exists$ it has a bound
then $|z|^n < M$
$\forall n \in \mathbb{N}$
then let n_0 be s.t
$M - (|z|-1) < |z|^{n_0} < M$
as : $|z|^{n_0+1} = |z|(|z|^{n_0})$
$= x + |z|^{n_0}$
$(|z|-1)|z|^{n_0}$

$\Rightarrow M < |z|^{n_0+1}$ $\ast$

(b) $z_n \to 0$
c_n is bounded
then $c_n z_n \to 0$
① $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t
$|z_n - 0| < \varepsilon$
$\forall n \geqslant n_0$
② and $\exists M$ s.t
$-M \leqslant c_n \leqslant M$ $\forall n \in \mathbb{N}$

now $|c_n z_n - 0| \leqslant |c_n||z_n| \leqslant |M||z_n|$
$\forall n \in \mathbb{N}$
$\Rightarrow < |M| \varepsilon$ $\forall n \geqslant n_0$

(c) $\frac{z^n}{n!} \to 0$ proof $\forall z \in \mathbb{C}$

here i.e now

first $\exists N \in \mathbb{N}$ s.t

$N > 2|z|$

now $\Rightarrow |z| < \frac{N}{2}$ some Number in $\mathbb{N}$

also note

$$0 \leqslant \left|\frac{z^n}{n!}\right| = \frac{|z|^N |z|^{n-N}}{n!} = \frac{|z|^N}{N!} \frac{|z|^{n-N}}{(N+1)(N+2)\cdots n}$$

$$\leqslant \left(\frac{|z|^N}{N!}\right)\left(\frac{N}{2(N+1)}\right)\cdots\cdots$$

$$\leqslant \frac{|z|^N}{N!}\left(\frac{1}{2}\right)^{n-N}$$

as $\left(\frac{1}{2}\right)^{n-N} \to 0$

$$\Rightarrow \frac{|z|^n}{n!} \to 0$$

$$\Rightarrow \left|\frac{z^n}{n!} - 0\right| < \varepsilon$$

$$\Rightarrow \frac{z^n}{n!} \to 0$$

(d) Here $|a_n - 0| = |\sqrt{n^2+2} - n|$

$$= \left|\frac{\cancel{n^2}+2-\cancel{n^2}}{\sqrt{n^2+2}+n}\right| = \frac{2}{\sqrt{n^2+2}+n} \leqslant \frac{\cancel{2}}{\cancel{2}n} = \frac{1}{n}$$

$$0 \leqslant |a_n| \leqslant \frac{1}{4}$$

$\therefore a_n \to 0$

4.2 $a_{n+2} = \frac{a_{n+1}+a_n}{2}$

$a_{n+1} = \frac{a_n + a_{n-1}}{2}$ $\Big\}$ $a_{n+2} - a_{n+1} = \frac{a_{n+1} - a_{n-1}}{2}$

$$= \frac{\frac{a_n + a_{n-1}}{2} - a_{n-1}}{2}$$

$$= \frac{a_n}{4} + \frac{a_{n-1}}{4} - \frac{a_{n-1}}{2}$$

$$a_{n+2} - a_{n+1} = \frac{1}{4}(a_n - a_{n-1})$$

$$a_{n+2} - a_{n+1} = \frac{1}{2}(a_{n+1} - a_{n-1})$$

$$a_{n+2} - a_{n+1} = \left(\frac{1}{2}\right)^n (k)$$

that mean seq is cauchy

$$a_{n+2} - a_{n+1} = \left(\frac{a_{n+1} + a_n}{2}\right) - a_{n+1}$$

$$\cancel{a_{n+2}} - \cancel{a_{n+1}} = \cancel{\frac{a_n}{2}} - \cancel{\frac{a_{n+1}}{2}}$$

$$\cancel{a_{n+1}} - \cancel{a_n} = \cancel{\frac{a_{n-1}}{2}} - \cancel{\frac{a_n}{2}}$$

$$\cancel{a_n} - \cancel{a_{n-1}} = \cancel{\frac{a_{n-2}}{2}} - \cancel{\frac{a_{n-1}}{2}}$$

$$\vdots$$

$$\cancel{a_3} - a_2 = \frac{a_1}{2} - \cancel{\frac{a_2}{2}}$$

$$a - a_2 = \frac{a_1}{2} - \frac{a}{2}$$

$$\frac{3a}{2} = \frac{a_1}{2} + a_2$$

$$\cancel{3}a = \frac{a_1 + 2a_2}{3}$$

$$a = \frac{a_1 + 2a_2}{3}$$

4.36 ($\Rightarrow$) given metric space is disconnected, then $\exists A, B \subseteq S$ s.t

$A \cap \bar{B} = \phi$ and
$B \cap \bar{A} = \phi$ where $A \cup B = S$
where $A, B \neq \phi$

Now, $A \cap B = \phi$ also $A \cup B = S$
$\Rightarrow A^c = B$
Now then if A is closed then A is open and vica versa.

($\Leftarrow$) A is both closed and open, then
$A^c = S - A$ is also closed and open
$B = S - A$
now $A \cup B = S$
and $A \cap B = \phi$
as B is both closed and open
$A \cap \bar{B} = \phi$
and $\bar{A} \cap B = \phi$
$\therefore$ disconnected.

4.37 last question $A \Leftrightarrow B$
$\sim(A) \Leftrightarrow \sim(B)$
method of negation

Tutorial-4:

4.10 (a) $\lim_{h \to 0} |f(x+h) - f(x)| = 0$

now $\lim_{h\to 0} |f(x+h) - f(x-h)| = 0$

as $|f(x+h) - f(x-h)| \leq |f(x+h) - f(x)| + |f(x-h) - f(x)|$

as $\lim_{h\to 0} |f(x+h) - f(x)| = 0$

$\lim_{-h\to 0} |f(x-h) - f(x)| = 0$

$\therefore \lim_{h\to 0} |f(x+h) - f(x-h)| = 0$

(b) if $\lim_{h\to 0} |f(x+h) - f(x-h)| = 0$

i.e let $f(x) = \begin{cases} 0 & x \neq 0 \\ 2 & x = 0 \end{cases}$

then $f(x+h) - f(x-h)$

$= h - (-h) = 2h$

as $\lim_{h\to 0} 2h = 0$

then

also, for $\lim_{h\to 0} |f(x+h) - f(x)|$ for $x = 0$

$= \lim_{h\to 0} |h - 2| = 2 \neq 0$

4.13 f is cont then $\lim_{h\to 0} f(x+h)$ exist and is equal to $f(x)$

also this mean that for some $\{x_n\}_{n=1}^{\infty}$ when

$x_n \to r \in \mathbb{R} \setminus \mathbb{Q}$

when x_n are in $\mathbb{Q}$

then also f is cont

$\lim_{x_n \to r} f(x_n) = 0 = f(r)$ (as $f(x)$ is cont)

rationales

$f(r) = 0$

for any $r \in \mathbb{R} \setminus \mathbb{Q}$

$\therefore \forall r \in \mathbb{R} \ f(r) = 0$

4.16 $f(x) = \begin{cases} 0 ; & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ 1 ; & x \in \mathbb{Q} \cap [0,1] \end{cases}$

now, for $f(x)$ to be cont
$\lim_{x \to r} f(x) = f(r)$ $\left(\because \lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n) \right)$

now, if $\{x_n\}_{n=1}^{\infty}$ is a seq in $\mathbb{Q} \cap [0,1]$
that converge to some $r \in (\mathbb{R} - \mathbb{Q}) \cap [0,1]$

then, if $f(x)$ is cont
then $\lim_{n \to \infty} f(x_n) = f(r)$
but
$f(x_n) = 0$
$\forall n$
and $f(r) = 1$

$\therefore$ Not cont at r, as this is general case,
f is not cont at every irrational
(similar for rational)

$g(x) = \begin{cases} 0 ; & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ x ; & x \in \mathbb{Q} \cap [0,1] \end{cases}$

for $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{Q} \cap [0,1]$
if $x_n \to x$ $x \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$
then if g is cont, then
$\lim_{n \to \infty} g(x_n) = g(\lim_{n \to \infty} x_n)$
$x = 0$ $*$
similarly for irrational $0 = x$ $*$

$x = 0$ (only for $x = 0$)
$\therefore$ cont at 0.

$h(x) = \begin{cases} 0 ; & x \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1] \\ 1/n ; & x \in \mathbb{Q} \cap [0,1] \\ & \text{where } x = m/n \\ & \text{s.t } \gcd(m,n) = 1 \\ 1 ; & x = 0 \end{cases}$

now first let's see for rationals what is happening. If we have
$\{x_n\}_{n=1}^{\infty}$ in $\mathbb{Q} \cap [0,1]$
s.t $\{x_n\}_{n=1}^{\infty} = \frac{1}{n}$ converges to 0
but $\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n) = f(0) = 1$
$0 = 1$ $*$
$\therefore$ Not cont at 0

also, for $\{x_n\} \to n$ where n is irrational
$\hookrightarrow$ in $\mathbb{Q}$

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = \frac{1}{n'}$$

$$0 = \frac{1}{n'}, \; *$$

$\therefore$ not-cont at every $x \in \mathbb{Q} \cap [0,1]$

now, for irrational let there be $\varepsilon > 0$
then $\exists\; N \leq \frac{1}{\varepsilon}$
largest positive number

then $T = \begin{cases} \{x \mid h(x) = 1\} \cup \{x \mid h(x) = \frac{1}{2}\} \ldots \cup \{x \mid h(x) = \frac{1}{N}\} & \varepsilon \leq 1 \\ \phi \text{ if } \varepsilon > 1 \end{cases}$

here T is finite. This means that we can find δ s.t

$|x - a| < \delta$
$\Rightarrow$ $x \in (a - \delta, a + \delta)$ contain no points in T
for small δ if $a \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1]$
then for $|x - a| < \delta$
$\Rightarrow$ $h(x) = 0$

$\therefore \forall \varepsilon > 0, \exists \delta > 0$ s.t
$|x - a| < \delta \Rightarrow |h(x) - 0| < \varepsilon$
for $x, a \in \mathbb{R} \setminus \mathbb{Q}$ as $h(x) = 0$
$0 < \varepsilon$ (true)

4.18 Now $f(x+y) = f(x) + f(y)$
and let $\{x\} \to r$
and
$y \to x_0$
then, where $y = x - r + x_0$

$$\lim_{x\to r} f(x) = \lim_{x\to r} f(y + r - x_0)$$

$$= \lim_{x\to r} f(y - x_0) + f(r)$$

$$\lim_{x\to r} f(x) = \lim_{y\to x_0} f(y - x_0) + f(r)$$

$$= \lim_{y\to x_0} \underbrace{\cancel{f(y)} - \cancel{f(x_0)}}_{\text{cont at } x_0} + f(r)$$

$$\lim_{x\to r} f(x) = f(r)$$

$\therefore$ cont at every $\mathbb{R}$.

Now, let $f(1) = f(\underbrace{\frac{1}{m} + \frac{1}{m} + \cdots \frac{1}{m}}_{m \text{ times for}})$
$m \in \mathbb{Q}$

then $f(1) = m f\left(\frac{1}{m}\right)$

$$\Rightarrow \frac{1}{m} f(1) = f\left(\frac{1}{m}\right)$$

also, $f(n/m) = f\left(\underbrace{\frac{1}{m} + \frac{1}{m} + \cdots \frac{1}{m}}_{n \text{ times}}\right) = n f\left(\frac{1}{m}\right) = \frac{n}{m} f(1)$

$$\therefore f\left(\frac{n}{m}\right) = \frac{n}{m} f(1)$$

$$\text{or } f(x) = x f(1) \quad \forall x \in \mathbb{Q}$$

now as f is cont over $\mathbb{R}$.

$\{x_n\} \to x \in \mathbb{R}$

where $\{x_n\} \in \mathbb{Q}$

then $\lim_{n\to\infty} f(x_n) = f(x)$

also $\lim_{n\to\infty} f(x_n) = f(x)$

$\lim_{n\to\infty} x_n f(1) = f(x)$

$\Rightarrow \quad x f(1) = f(x)$

$\forall x \in \mathbb{R}$

$\therefore a = f(1)$

4.21 $f: S \to R$ be cont on an open set in $\mathbb{R}^n$. $P \in S$ and $f(P) > 0$

T.S.T: $\exists$ n-ball $B(P, r)$ s.t $f(x) > 0 \ \forall x$ in $B(P, r)$

$f: S \to R$ is cont on all open sets in $\mathbb{R}^n$

so, if S is open then

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t

for $B(x, \delta) \subseteq S$

$f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$

as S is open and $P \in S$

$\exists \ B(P, \delta) \subseteq S$

and as $f(P) > 0$

and as

$f(B(P, \delta)) \subseteq B(f(P), \varepsilon)$

$\forall x \in B(P, \delta)$

$f(x) \in B(f(P), \varepsilon)$

now as $f(P) > 0$

if $\varepsilon = f(P)/2$

then

$\forall y$ in $B(f(P), f(P)/2)$

will be

s.t $d(f(P), y) < f(P)/2$

$\|f(P) - y\| < \|f(P)/2\|$

$\|f(P)\| - \|y\| < \frac{\|f(P)\|}{2}$

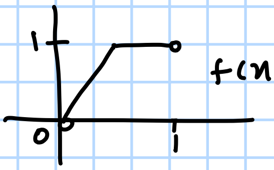
$\|y\| \geq \frac{\|f(P)\|}{2}$

$\therefore y > 0$, so $\forall x \in B(P, \delta) \Rightarrow f(x) \in B(f(P), \varepsilon)$

for $\varepsilon = f(P)/2$ $\quad f(x) > 0$

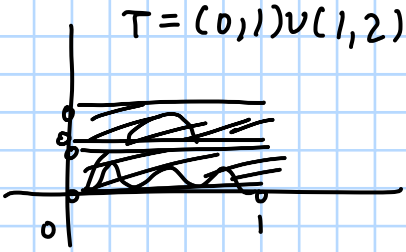
4.28 real valued f, cont on S s.t $f(S)=T$

(a) $S=(0,1)$
$T=(0,1]$



$f(x)=\begin{cases} 2x \,; & x\in(0,\frac{1}{2}] \\ 1 \,; & x\in(\frac{1}{2},1) \end{cases}$

(b) $S=(0,1)$ $\quad T=(0,1)\cup(1,2)$



or as S is connected
T is not connected

for a f to be cont in $(0,1)\cup(1,2)$ and surjective
there should be atleast one $x_1\in S$ s.t
$f(x_1)\in(1,2)$
and $f(x_2)\in(0,1)$
for $x_2\in S$

but this means that for f to be cont
$|x_2-x_1|<\delta$
for $\forall\,\varepsilon>0$
$|f(x_2)-f(x_1)|<\varepsilon$
for $x_2=x_1$, $f(x)=1$
but as
1 is not
in T
$f(x)$ is not continuous.

(c) $S=\mathbb{R}$, $T=$ set of rationals : No
similar to previous case

or argue with compactness

(d) yes $S=[0,1]\cup[2,3]$
$T=\{0,1\}$

then $f(x)=\begin{cases} 0 \,; & x\in[0,1] \\ 1 \,; & x\in[2,3] \end{cases}$

(e) $S=[0,1]\times[0,1]$
$T=\mathbb{R}^2$

doubt

(f) $S=[0,1]\times[0,1]$
$T=(0,1)\times(0,1)$
same as above, V open then $f^{-1}(V)$ also open
here $V=T$ is open but
$S=f^{-1}(V)$ not open

(g) $S=(0,1)\times(0,1)$
$T=\mathbb{R}^2$
for $f(x,y)=(\cot\pi x, \cot\pi y)$

4.30 To prove : f is cont $\Leftrightarrow f(\bar{A}) \subseteq \overline{f(A)}$

proof : ($\Rightarrow$) if f is cont then t.s.t $f(\bar{A}) \subseteq \overline{f(A)}$

then $f(A) \subseteq \overline{f(A)}$

also as $A \subseteq f^{-1} f(A)$

$\Rightarrow A \subseteq f^{-1}(\overline{f(A)})$

now $\bar{A} \subseteq f^{-1}(\overline{f(A)})$

but as $\overline{f(A)}$ is closed

$f^{-1}(\overline{f(A)})$ is also closed

$\therefore \bar{A} \subseteq f^{-1} \overline{f(A)}$

now, $f(\bar{A}) \subseteq f f^{-1} \overline{f(A)}$

$\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$

($\Leftarrow$) for the converse, if we prove that $\forall$ V closed $f^{-1}(V)$ is also closed then we are done.

given every subset of S (A) s.t $f(\bar{A}) \subseteq \overline{f(A)}$

now, let C = closed subset of T s.t $f^{-1}(C) = A$

then

$f(\bar{A}) = f(\overline{f^{-1}(C)}) \subseteq \overline{f(A)}$

$= \overline{f(f^{-1}(C))}$

$f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))} = \bar{C}$

$\Rightarrow f(\overline{f^{-1}(C)}) \subseteq \bar{C} = C$ as C is closed

$\Rightarrow f(\overline{f^{-1}(C)}) \subseteq C$

i.e $f(\bar{A}) \subseteq C$

also $f(\bar{A}) \subseteq \overline{f(A)}$

$\bar{A} \subseteq f^{-1} \overline{f(A)}$

$\Rightarrow \bar{A} \subseteq f^{-1}(\overline{f(f^{-1}(C))})$

$= f^{-1}(\bar{C})$

$= f^{-1}(C)$

$= A$

i.e $\bar{A} \subseteq A$

or A is closed

$\therefore f^{-1}(C)$ is closed, given C is closed

$\Rightarrow f$ is cont.

4.33 let $S = (0,1]$ and $\{x_n\} = \frac{1}{n}$

now $\{x_n\}_{n=1}^{\infty}$ is cauchy and also

$f(x) = \frac{1}{x}$

so $f(x_n) = n$ is not cauchy in $\mathbb{R}$

when $f: S \to \mathbb{R}$

Tutorial 6 :

4.51 $f(x) = x^2$ in $\mathbb{R}$

lets say $f(x)$ is uniformly cont on $\mathbb{R}$

then

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\forall x, p \in \mathbb{R}$

s.t

$|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

for $\varepsilon = 1$ i.f there exist δ s.t

$|x - p| < \delta \Rightarrow |x^2 - p^2| < 1$

$\Rightarrow |x - p||x + p| < 1$

$\Rightarrow |x + p| < \frac{1}{\delta}$

$|x + p| < \frac{1}{\delta}$ should be true for all x, p s.t $|x - p| < \delta$

but for $x = 0.5/\delta + \delta/2$

$y = 0.5/\delta$

as $|x - y| < \delta$

but $|x + y| = \frac{1}{\delta} + \delta/2$

$> \frac{1}{\delta}$

$\therefore$ contradiction

$\therefore f$ is not uniformly cont over $\mathbb{R}$

4.53 $f \leftarrow$ function defined on set S in $\mathbb{R}^n$

$f(S) \subseteq \mathbb{R}^m$

$f : S \longrightarrow \mathbb{R}^m$

s.t $f(S) \subseteq \mathbb{R}^m$

$g : f(S) \longrightarrow \mathbb{R}^k$

$h(x) = (g \circ f)(x) : S \longrightarrow \mathbb{R}^k$

uniformly cont on S

uniformly cont on $f(S)$

as f is uniformly cont on S

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\forall x, p \in S$

where $|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

also as g is uniformly cont on $f(S)$

$\forall \varepsilon' > 0, \exists \delta' > 0$ s.t

$\forall f(x), f(p) \in f(S)$

$|f(x) - f(p)| < \delta'$

$\Rightarrow |g(f(x)) - g(f(p))| < \varepsilon'$

now to show that h is uniformly cont, we have

To prove : $\forall \varepsilon'' > 0, \exists \delta'' > 0$ s.t $\forall x, p \in S$
s.t $|x-p| < \delta''$
$\Rightarrow |g(f(x)) - g(f(p))| < \varepsilon''$

as $\forall \varepsilon' > 0, \exists \delta' > 0$ s.t
$|g(f(x)) - g(f(p))| < \varepsilon'$ — ①
$\forall f(x), f(p) \in f(S)$ s.t
$|f(x) - f(p)| < \delta'$ — ②

putting $\varepsilon = \delta'$ we get, $\exists \delta > 0$ s.t
$|f(x) - f(p)| < \delta'$
$\forall x, p \in S$ s.t
$|x - p| < \delta$ — ③

$\therefore \forall \varepsilon' > 0, \exists \delta > 0$ s.t
$|h(x) - h(p)| < \varepsilon'$
$\forall x, p \in S$ s.t
$|x - p| < \delta$
$\therefore$ h is uniformly cont.

4.54 $f : S \to T$
uniformly cont. on S
where S and T are metric spaces.

To prove : $\{x_n\}$ is cauchy in S $\Rightarrow \{f(x_n)\}$ is cauchy in T.

proof : as $\{x_n\}$ is cauchy and f is unif. cont
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t $|x_n - x_m| < \varepsilon \; \forall n, m \geq n_0$
and $\forall \varepsilon' > 0, \exists \delta > 0$ s.t $\forall x, p \in S$
s.t $|x - p| < \delta$
$\Rightarrow |f(x) - f(p)| < \varepsilon'$

now, putting $\varepsilon = \delta$ we get :
$\forall \varepsilon' > 0, \exists \delta > 0$ and $\exists n_0 > 0$ s.t
$|x_n - x_m| < \delta$
$\forall n, m \geq n_0$
$\Rightarrow |f(x_n) - f(x_m)| < \varepsilon'$

$\therefore$ cauchy

or $\forall \varepsilon' > 0, \exists n_0 > 0$ s.t
$|f(x_n) - f(x_m)| < \varepsilon' \; \forall m, n \geq n_0$

4.56 $(S,d) \leftarrow$ metric space

$\underset{\substack{\uparrow \\ \text{non-empty subset of } S}}{A} \subseteq S$ s.t $A \neq \phi$

$f_A : S \longrightarrow \mathbb{R}$

$f_A(x) = \inf\{ d(x,y) \mid \forall\, y \in A \}$

$\downarrow$

or the least distance of x from A.

(a) To prove: f_A is uniformly cont on S

proof:

To show that f is uniformly cont.
we have to show that

$\forall \varepsilon > 0, \ \exists \delta > 0$ s.t
$d(x_1, x_2) < \delta$ for $x_1, x_2 \in S$
then $|f_A(x_1) - f_A(x_2)| < \varepsilon$

$d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$
& $d(x_2, y) \leq d(x_1, x_2) + d(x_1, y)$

$\inf\{ d(x_1, y) \mid y \in A\} \leq d(x_1, x_2) + \inf\{ d(x_2, y) \mid y \in A\}$

$\inf\{ d(x_2, y) \mid y \in A\} \leq d(x_1, x_2) + \inf\{ d(x_2, y) \mid y \in A\}$

$\Rightarrow |f_A(x_1) - f_A(x_2)| \leq d(x_1, x_2)$

for $\delta = \varepsilon$ we have $d(x_1, x_2) < \delta = \varepsilon$

$\therefore f_A$ is uniformly cont on S.

(b) To prove $cl(A) = \{ x \mid x \in S$ and $f_A(x) = 0 \}$

$K = \{ x \mid x \in S$ and $f_A(x) = 0 \}$

if $x \in cl(A)$ then
x is a limit point of A
or i.e $\forall r > 0$
$(B(x, r) - \{x\}) \cap A \neq \phi$

let $y_k \in (B(x, 1/k) - \{x\}) \cap A$

then as $y_k \longrightarrow x$
as $k \longrightarrow \infty$

$\inf\{ d(x,y) \mid y \in A\} \leq d(x, y_k)$
as $d(x, y_k) \longrightarrow 0$
as $k \longrightarrow \infty$

by sandwich theorem

$\inf\{ d(x,y) \mid y \in A\} = 0$
or $x \in cl(A) \Rightarrow x \in K$

$cl(A) \subseteq K$

now if $x \in K$ then $f_A(x) = \inf\{d(x,y) \mid y \in A\} = 0$

or $\inf\{d(x,y) \mid y \in A\} = 0$

as $\inf\{d(x,y) \mid y \in A\} = 0$

$\forall \varepsilon_k > 0$, $\exists$ $y_k \in A$ s.t $d(x, y_k) < \varepsilon_k$

or $(B(x, \varepsilon_k) - \{x\}) \cap A \neq \phi$

as $y_k \in B(x, \varepsilon_k) - \{x\}$ and

$y_k \in A$

so, $x \in cl(A)$ or $K = cl(A)$

4.58 f defined on $\mathbb{R}^1$ by:

(a) $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

as $\sin x$ cont for all $x \in \mathbb{R}$ and $1/x$ cont for all $x \in \mathbb{R} \setminus \{0\}$

$f(x) = \frac{\sin x}{x}$ is cont for all $x \in \mathbb{R} \setminus \{0\}$

$\lim_{x \to 0} f(x) = 1 \neq f(0)$

so <u>removable discontinuity</u>

(b) $f(x) = e^{1/x}$ $x \neq 0$
$= 0$ $x = 0$

then cont $\forall x \in \mathbb{R} \setminus \{0\}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{1/x}$

$\lim_{x \to 0^+} e^{1/x} = \infty$

$\lim_{x \to 0^-} e^{1/x} =$

$f(0) = 0$ <u>irremovable</u>

(c) $f(x) = e^{1/x} + \sin \frac{1}{x}$ same <u>irremovable</u> as

$x \to 0^+$ ∞

$x \to 0^-$ undefined and $f(0) = 0$

(d) $f(x) = \dfrac{1}{1-e^{1/x}}$ $\quad x \neq 0$
$f(0) = 0$

$1 = e^{1/x}$ at
$1/x = \log_e 1 = 0$

or $x \to \infty$

$$\lim_{x \to 0^+} \frac{1}{1-e^{1/x}} = 0$$
$$\lim_{x \to 0^-} \frac{1}{1-e^{1/x}} = 1$$

$f(0) = 0$

or $\underline{\text{imemovable}}$

8.7 $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$

(a) $\cos n$

$\overline{\lim}\, a_n = l$
if $\exists\, l \in \mathbb{R}$ s.t
$\forall \varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t

$a_n < l + \varepsilon \quad \forall n \geq n_0$

and $\forall m \in \mathbb{N}$, $\exists\, n_1 \in \mathbb{N}$
s.t $a_{n_1} > l - \varepsilon$

and $\underline{\lim}\, a_n = -\overline{\lim}\,(-a_n)$

now, for $\cos n$
say $\overline{\lim} \cos n = 1$

then $\forall \varepsilon > 0$ as $\cos n < 1$
$\cos n < 1 + \varepsilon$
$\therefore \forall n \in \mathbb{N}$ $\cos n < 1 + \varepsilon$

also $\forall \varepsilon > 0$, and $\forall m > 0$

as $\forall n > m$
$\cos(n) \in [-1, 1]$
for :
$\cos n > 1 - \varepsilon$
as $\varepsilon > 0$
$1 - \varepsilon < 1$
so $\exists$ some $n > m$ s.t
$\cos(n) > 1 - \varepsilon$
as $\cos(n) \in (-1, 1)$

now $\underline{\lim} \cos n = -\overline{\lim} \cos(-n)$
$= -\overline{\lim} \cos(n)$
$= -1$

(b) $\left(1+\frac{1}{n}\right)\cos \pi n = a_n$

say $\overline{\lim}\, a_n = 1$

$\underline{\lim}\, a_n = -1$

(c) $n \sin \frac{n\pi}{3}$ $\quad$ $\overline{\lim}\, a_n \to \infty$
(unbounded)
$\underline{\lim}\, a_n \to -\infty$

(d) $\sin \frac{n\pi}{2} \cos \frac{n\pi}{2}$

$$\overline{\lim}\, \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} = \overline{\lim}\, \frac{1}{2} \sin n\pi$$
$$= \overline{\lim}\, (0)$$

or $\overline{\lim}\, a_n = 0$
$\underline{\lim}\, a_n = 0$

(e) $(-1)^n n / (1+n)^n$

$$\overline{\lim}\, (-1)^n n / (1+n)^n$$
$$= \overline{\lim}\, \frac{n}{(1+n)^n} = 0$$

$$\underline{\lim}\, (-1)^n n / (1+n)^n$$
$$= 0$$

(f) $\frac{n}{3} - \left[\frac{n}{3}\right]$

for $n = 3k$
$n/3 - [n/3] = 0$
$n = 3k+1$
$\cancel{k} + 1/3 - \cancel{k} = 1/3$
$n = 3k+2$
$k + 2/3 - k = 2/3$

or $\overline{\lim}\, a_n = 2/3$
$\underline{\lim}\, a_n = 0$

8.9 $|a_n| < 2$
$|a_{n+2} - a_{n+1}| \leq \frac{1}{8} |a_{n+1}^2 - a_n^2|$
$= \frac{1}{8} |a_{n+1} - a_n| |a_{n+1} + a_n| \leq \frac{1}{2} |a_{n+1} - a_n|$

$|a_{n+1} - a_n| \leqslant \left(\frac{1}{2}\right)^{n-3}$

$|a_{n+k} - a_n| \leqslant \Sigma\,|a_{n+j} - a_{n+j-1}|$

$\leqslant \Sigma\left(\frac{1}{2}\right)^{n+j-4}$

$\leqslant \left(\frac{1}{2}\right)^{n-2} \longrightarrow \infty$

ces $n \longrightarrow \infty$

or $\{a_n\}$ is cauy in $\mathbb{R}$

or $\{a_n\}$ conv.

Tutorial-7:

8.15

a) $\sum_{n=1}^{\infty} n^3 e^{-n} = \sum_{n=1}^{\infty} a_n$

now

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3 e^{-n-1}}{(n)^3 e^{-n}}\right|$$

$$= \left|\left(1+\frac{1}{n}\right)^3 \frac{1}{e}\right|$$

now $\lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^3 \frac{1}{e}\right| = \frac{1}{e}$ $\left(\because \lim_{n\to\infty} \frac{1}{n} = 0\right)$

($\because$ Algebra of limits)

so $\overline{\lim} \left|\left(1+\frac{1}{n}\right)^3 \frac{1}{e}\right| = \frac{1}{e}$

and

$\underline{\lim} \left|\left(1+\frac{1}{n}\right)^3 \frac{1}{e}\right| = \frac{1}{e}$

using ratio test

$r = 1/e \qquad R = 1/e$

$= \underline{\lim} \left|\frac{a_{n+1}}{a_n}\right| \qquad = \overline{\lim} \left|\frac{a_{n+1}}{a_n}\right|$

as $R < 1$, the series converges

(c) $\sum_{n=1}^{\infty} p^n n^p = \sum_{n=1}^{\infty} a_n$

$(p > 0)$

where $a_n = p^n n^p$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{p^{n+1}}{p^n} \times \frac{(n+1)^p}{(n)^p}\right|$$

$$= \left|p \times \left(1+\frac{1}{n}\right)^p\right|$$

now as $p > 0$

$\lim_{n\to\infty} \frac{1}{n} = 0$ then

because of algebra of limits

$\lim_{n\to\infty} \left|p \times \left(1+\frac{1}{n}\right)^p\right| = |p|$

as $p > 0$

$\lim_{n\to\infty} \left|p \times \left(1+\frac{1}{n}\right)^p\right| = p$

$\therefore \overline{\lim} \left|\frac{a_{n+1}}{a_n}\right| = p = \underline{\lim} \left|\frac{a_{n+1}}{a_n}\right|$

two cases:

Case I: $p<1$, then $\lim \left|\frac{a_{n+1}}{a_n}\right| = p < 1$

or the series converges using the ratio test

Case II: $p > 1$, then $\lim \left|\frac{a_{n+1}}{a_n}\right| = p > 1$

or the series diverges using ratio test.

Case III: $p=1$, then $\sum_{n=1}^{\infty} p^n (n)^p = \sum_{n=1}^{\infty} (1)^n (n)^1$

$= \sum_{n=1}^{\infty} n$

as $\sum_{n=1}^{\infty} n$, we have

$\lim_{n\to\infty} a_n = \infty \quad (\because a_n = n)$

or $\forall p \in \mathbb{N}$

$|a_n + a_{n+1} + \cdots a_{n+p}| \longrightarrow \infty$ as $n \to \infty$

$\therefore$ Cauchy comparison test fails

$\therefore$ for $p=1$, the series diverges.

$\therefore$ $p \in (0,1) \to$ converges

$p \in [1,\infty) \to$ diverges

(d) $\sum_{n=2}^{\infty} \frac{1}{n^p - n^q} \quad (0<q<p)$

then $\frac{1}{n^p - n^q} = \frac{1}{n^p(1-n^{q-p})}$

as $q<p$ n^{q-p}, $q-p<0$

or $\frac{1}{n^{p-q}} > 0$

so $\frac{1}{n^p - n^q} = \frac{1}{n^p} \times \frac{1}{\left(1 - \frac{1}{n^{p-q}}\right)}$

now, using limit comparison test with

$a_n = \frac{1}{n^p} \times \frac{1}{\left(1-\frac{1}{n^{p-q}}\right)}$

$b_n = \frac{1}{n^p}$

Now using theorm 3.10 $\left(\sum \frac{1}{n^p} \begin{matrix} \text{conv for } p>1 \\ \text{div for } p\le 1 \end{matrix} \right)$

b_n converges for $p>1$
diverges for $p \le 1$

limit comparison test:

<u>case I) $p>1$:</u>

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cancel{\frac{1}{n^p}} \times \frac{1}{\left(1 - \frac{1}{(n)^{p-a}}\right)}}{\cancel{\frac{1}{n^p}}} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{(n)^{p-a}}}$$

as $\lim_{n\to\infty} \frac{1}{n} = 0$

using algebra of limits

$$\lim_{n\to\infty} \frac{1}{1-\frac{1}{(n)^{p-a}}} = 1$$

as $0 < \lim_{n\to\infty} \frac{1}{1-\frac{1}{(n)^{p-a}}} = 1 < \infty$

and as $p>1$, $\sum b_n$ converges by thm 3.10

using limit comparison test, $\sum a_n$ conv. for $p>1$

<u>case II) $p \le 1$:</u> $(p>0)$

$$0 < \lim_{n\to\infty} \frac{1}{1-\left(\frac{1}{n^{p-a}}\right)} = 1 \quad (\text{same as above}) < \infty$$

but as $\sum b_n$ diverges for $p \in (0,1]$

$\sum a_n$ also diverge (limit comparison test)

(e) $\sum_{n=1}^{\infty} n^{-1-1/n} = \sum_{n=1}^{\infty} a_n \qquad a_n = n^{-1-1/n}$

let $b_n = \frac{1}{n}$

(note using $p=1$, $\sum b_n$ diverges)

first let's find this lim:

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = e$$

then $\log(e) = \frac{1}{n} \log\left(\frac{1}{n}\right)$

let $\frac{1}{n} = x$ as $n\to\infty$, $x \to 0^+$

then $x\log(x)$

or $\lim_{x\to 0^+} x\log(x)$

now, $x\log(x)$

$$\frac{d}{dx}\left(x\log(x)\right) = \log(x)+1$$

as $\log(x)+1 < 0$

for all e^{-n} s.t $n \geq 1$ and $n \in \mathbb{N}$

then on $(0, e^{-n})$ $x\log x$ is dec

as $x\log x < 0$ for $(0, e^{-n})$

and

$$x\log x > (e^{-n})\log(e^{-n})$$

$$0 > x\log x > (-n)(e^{-n})$$

by Sandwich theorem

$$\lim_{x\to 0^+} x\log x = 0$$

or $\lim_{n\to\infty}\left(\frac{1}{n}\right)\log\left(\frac{1}{n}\right) = 0 = \log e$

then $\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{n}} = e^{\log e} = e^{0} = 1$

<u>Now</u>, $\lim_{n\to\infty} \frac{a_n}{b_n} = \log_{n\to\infty} \frac{\cancel{n+1}-\frac{1}{n}}{\cancel{n+}}$

$$= \log_{n\to\infty}\left(\frac{1}{n}\right)^{1/n} = 1 \quad \left(\text{just proved}\right)$$

so, by <u>limit comparison test</u> as

$\sum b_n$ diverges $\left(p=1, \text{ by theorm } 3.10\right)$

so $\sum a_n$ also diverges.

(f) $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$ $(0 < q < p)$

putting $a_n = \frac{1}{p^n - q^n}$

$b_n = \frac{1}{(p)^n}$

<u>Case I)</u> <u>$p > 1$</u> $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\frac{1}{p}\right)^n = 0$ (proved in previous test)

as $\left|\frac{1}{p}\right| < 1$

now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{p^n - q^n}\right)}{\left(\frac{1}{p^n}\right)}$

$$= \lim_{n\to\infty} \frac{1}{1-\left(\frac{q}{p}\right)^n}$$

as $q < p \Rightarrow \frac{q}{p} < 1$

or $\left(\frac{q}{p}\right)^n \to 0$ as $n \to \infty$

so $\lim_{n\to\infty} \frac{1}{1-\left(\frac{q}{p}\right)^n} = 1$

now as $0 < \lim_{n\to\infty} \frac{1}{1-\left(\frac{q}{p}\right)^n} < \infty$

as $\Sigma b_n = \Sigma \frac{1}{(p)^n}$

we have

$\left|\frac{b_{n+1}}{b_n}\right| = \frac{\frac{1}{(p)^{n+1}}}{\frac{1}{(p)^n}} = \frac{1}{p} < 1 \Rightarrow R < 1$ ($\because p > 1 \Rightarrow \frac{1}{p}$

$\overline{\lim} \left|\frac{b_{n+1}}{b_n}\right| = \frac{1}{p} = \underline{\lim} \left|\frac{b_{n+1}}{b_n}\right|$ (ratio test)

< 1

we have Σb_n to be conv

($\because \overline{\lim} \left|\frac{b_{n+1}}{b_n}\right| < 1$)

so using limit comparison test

Σa_n is conv for $p > 1$

case II) $p \leq 1$, similar to above Σb_n is div as $\underline{\lim} \left|\frac{b_{n+1}}{b_n}\right| = \frac{1}{p} > 1$

now using limit comparison test

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (similar to above)

as Σb_n div, Σa_n div for $p < 1$

Case III > we have $p=1$: (as $q<p<1 \Rightarrow q<1$)

$$\sum b_n = \sum \left(\frac{1}{p}\right)^n = \sum_n 1$$

as $\sum_{n=1} 1 = n$

and as $n \to \infty$

$$\sum_{n=1}^{n} 1 \to \infty$$

$\sum b_n$ is divg

Now $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1-(q)^n}$

as $q<1$

$(q)^n = 0$ for $n \to \infty$

so $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = 1$ from Algebra of limits

$\therefore$ By limit comparison test $\sum a_n$ is divg.

(g) $\sum_{n=1}^{\infty} \frac{1}{n \log(1+\frac{1}{n})} = \sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n\log(1+\frac{1}{n})}$

as $\frac{d}{dx} \log\left(1+\frac{1}{x}\right) = \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right) < 0$ $\forall x > 0$

we have for any n $\log(1+\frac{1}{n})$ as a dec function

or $\log(1+\frac{1}{n}) < \log(1+1)$

$$\Rightarrow \frac{1}{\log(1+\frac{1}{n})} > \frac{1}{\log(2)}$$

$$\Rightarrow \frac{1}{n\log(1+\frac{1}{n})} > \frac{1}{n\log(2)}$$

as $\sum \frac{1}{n}$ is a divg series or as $n \to \infty$ $\sum \frac{1}{n} \to \infty$

as $\frac{1}{\log(2)}$ is a const positive term

$$\sum \frac{1}{n} \times \frac{1}{\log(2)} \to \infty \text{ as } n \to \infty$$

now, as $\sum_{k=1}^{n} \frac{1}{k} \frac{1}{\log(2)} < \sum_{k=1}^{n} \frac{1}{k\log\left(1+\frac{1}{k}\right)}$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} \frac{1}{\log(2)} < \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k\log\left(1+\frac{1}{k}\right)}$$

as $\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{\log(2)} \to \infty$

we have $\sum_{n=1}^{\infty} \frac{1}{n\log\left(1+\frac{1}{n}\right)} \to \infty$

or $\sum_{n=1}^{\infty} \frac{1}{n\log\left(1+\frac{1}{n}\right)}$ diverges

8.24 $\sum a_n$ conv each $a_n > 0$

To prove : $\sum (a_n a_{n+1})^{1/2}$

as $\sum a_n$ converges

$\frac{a_n + a_{n+1}}{2} \geq (a_n a_{n+1})^{1/2}$ ($\because$ AP $\geq$ GP)

as $(a_n a_{n+1})^{1/2} \leq \frac{a_n + a_{n+1}}{2}$

as $\sum a_n$ converges, by comp comp test

$(a_n a_{n+1})^{1/2}$ also conv.

if $\{a_n\}$ is monotonic, then $a_n \geq a_{n+1}$
or $a_n \leq a_{n+1}$

case I > $(a_n a_{n+1})^{1/2} \geq a_n$

case II > $(a_n a_{n+1})^{1/2} \geq a_{n+1}$

$\therefore$ conv $\{a_n\}$ in both case

8.25 $\sum a_n$ conv ab.

(a) $\sum a_n^2$ also conv absolutely

now if $\sum a_n$ conv
$a_n \to 0$ as
$n \to \infty$

now, $\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t
$|a_n - 0| < \varepsilon \ \forall n \geq n_0$

now if $\varepsilon = 1$
then
$|a_n| < 1 \quad \forall n \geq n_0$

now $a_n^2 < |a_n| < 1 \quad \forall n \geq n_0$

so $a_n^2 < |a_n|$
as a_n conv absolutely
a_n^2 also converges
from comparison test

(b) $\sum \frac{a_n}{1+a_n}$ $(a_n \neq -1)$ we have $|a_n| < \varepsilon \ \forall n \geq n_0$
$-|a_n| > -\varepsilon$ (prev)

$|1+a_n| > 1 - |a_n| > 1/2$ for $\varepsilon = 1/2$

$$\Rightarrow |1+a_n| > 1/2$$

$$\Rightarrow \frac{1}{|1+a_n|} < 2$$

$$\Rightarrow \frac{|a_n|}{|1+a_n|} < 2|a_n|$$

by comparison test $\frac{|a_n|}{|1+a_n|}$ is conv

so $\sum \frac{a_n}{1+a_n}$ is abs. conv

(c) $\sum \frac{a_n^2}{1+a_n^2}$, as $\frac{a_n^2}{1+a_n^2} \leq a_n^2$

we have $\frac{a_n^2}{1+a_n^2}$ conv by comparison test.

Tutorial - 8 :

1. (a) $a \in \mathbb{R}$

f is twice diff on (a, ∞)

$M_0, M_1, M_2 \rightarrow$ least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ on (a, ∞)

To prove: $M_1^2 \leq 4 M_0 M_2$

proof: $\exists\, x_1$ b.w x and c s.t

$$f(x) = f(c) + \sum_{k=1}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n)}(x_1)(x-c)^n}{n!}$$

here as twice diff for $x = c+h$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(x_1)(x-c)^2}{2!}$$

$$f(c+h) = f(c) + f'(c)(h) + \frac{f''(x_1)}{2!}(h)^2$$

$$\frac{f(c+h) - f(c)}{h} = f'(c) + \frac{f''(x_1)(h)}{2!}$$

new x

for $x = c$ for some $x_1 \in (x-h, x+h)$

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \frac{f''(x_1)(h)}{2!}$$

$$f'(x) = \left[\frac{f(x+h) - f(x)}{h}\right] - \frac{f''(x_1)(h)}{2!}$$

$$|f'(x)| = \left|\frac{f(x+h) - f(x)}{h} - \frac{f''(x_1)(h)}{2!}\right|$$

$$\Rightarrow |f'(x)| \leq \left|\frac{f(x+h) - f(x)}{h}\right| + \left|\frac{f''(x_1)(h)}{2!}\right|$$

$$\Rightarrow |f'(x)| \leq \frac{2}{h} M_0 + \frac{h}{2} M_2$$

as $|f'(x)| \leq M_1$ and M_1 is l.u.B

$$\Rightarrow M_1 \leq \frac{2}{h} M_0 + \frac{h}{2} M_2$$

now to maximise/minimise right side let's put

$$\frac{\partial}{\partial h}\left(\frac{2}{h} M_0 + \frac{h}{2} M_2\right) = 0$$

$$\Rightarrow h = 2\sqrt{\frac{M_0}{M_2}} \quad (\text{for } M_0 \neq 0,\ M_2 \neq 0)$$

if $M_0 = 0$ or $M_2 = 0$ then we can make one none case and the last case both 0.

now $M_1 \leq \sqrt{M_2 M_0} + \sqrt{M_2 M_0}$

$\Rightarrow (M_1)^2 \leq 4 M_0 M_2$

(b) f is twice diff on $(0,\infty)$

f'' bounded

$f(x) \to 0$
as $x \to \infty$

To prove: $f'(x) \to 0$ as $x \to \infty$

now for (a,∞)

$M_1^2 \leq 4 M_2 M_0$

were as $f(x) \to 0$
as $x \to \infty$

$\forall \varepsilon > 0$, if $x \to \infty$

$|f(x) - 0| < \varepsilon$

or $|f(x)| < \varepsilon$
for $x \to \infty$

now as $|f''(x)|$ is bounded
say M_2
on $(0,\infty)$

for $x \to \infty$

$|f''(x)| \leq M_2$

then for $x \to \infty$

$M_1^2 \leq 4 M_2 \varepsilon$

$\Rightarrow |M_1| \leq 2\sqrt{M_2}\sqrt{\varepsilon} = \varepsilon'$

$\Rightarrow |M_1| \leq \varepsilon$

so as $x \to \infty$

$|M_1| \to 0$

or $|f'(x)| \leq |M_1|$

by sandwich theorm

$f'(x) \to 0$

5.1 Lipschitz condition of order α at c if $\exists M > 0$ (may depend on c) and a 1-Ball $B(c)$ s.t

$$|f(x) - f(c)| < M|x-c|^{\alpha}$$
where $x \in B(c)$, $x \neq c$

(a) <u>To prove</u>: cont at c if $\alpha > 0$
derivative at c if $\alpha > 1$

<u>proof</u>: as $|f(x) - f(c)| < M|x-c|^{\alpha}$
if $\alpha > 0$

$$|f(x) - f(c)| < M|x-c|^{\alpha}$$

as $x \longrightarrow c$
by sandwich theorem
$$f(x) \longrightarrow f(c)$$
$\therefore$ $f(x)$ is cont at c

if $\alpha > 1$ then
$$\left|\frac{f(x) - f(c)}{x-c}\right| < M(x-c)^{\alpha-1}$$
similar to above

as $x \to c$ $\quad \dfrac{f(x) - f(c)}{x-c} \longrightarrow 0$

or
$$f'(c) = 0$$

<u>Note</u>: we can also use $\delta \leq (\varepsilon/M)^{1/\alpha}$

then $|f(x) - f(c)| < M\delta^{\alpha} = \varepsilon$

(b) Example of f satisfying Lipschitz condition of order 1 at c but $f'(c)$ does not exist.

$f(x) = |x|$ at $c = 0$

$$||x| - 0| = |x| < M|x|^{1}$$
for $M > 1$ (any)

now, as $\displaystyle\lim_{x \to c^+} \frac{f(x) - f(c)}{x-c} = 1$

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x-c} = -1$$

$f'(c)$ does not exist

5.2 To find: interval of f inc or decreasing and maxima / minima

a) $f(x) = x^3 + ax + b$

$f'(x) = 3x^2 + a = 0$

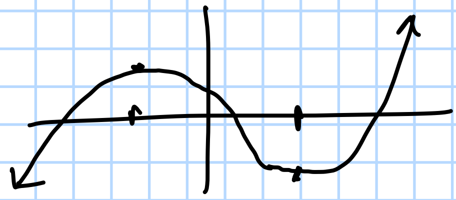
$a = -3x^2$

$x = \pm\sqrt{\frac{-a}{3}} \quad (a<0)$

for $x \in \left(-\infty, -\sqrt{\frac{-a}{3}}\right)$

$f'(x) > 0$ (inc)

$x \in \left(-\sqrt{\frac{-a}{3}}, \sqrt{\frac{-a}{3}}\right)$



$f'(x) < 0$ (dec)

$x \in \left(\sqrt{\frac{-a}{3}}, \infty\right)$

$f'(x) > 0$ (inc)

so in those intervals, also

$$f\left(-\sqrt{\frac{-a}{3}}\right) = \left(-\sqrt{\frac{-a}{3}}\right)^3 + a\left(-\sqrt{\frac{-a}{3}}\right) + b$$

$\downarrow$

local maxima

$$f\left(+\sqrt{\frac{-a}{3}}\right) = \left(+\sqrt{\frac{-a}{3}}\right)^3 + a\left(\sqrt{\frac{-a}{3}}\right) + b$$

$\downarrow$

local minima

<u>Note</u>: if $a > 0$ then cont inc function from $(-\infty, \infty)$ as no minima/maxima

(b) $f(x) = \log(x^2 - 9)$ $\quad |x| > 3$ or $x \in (-\infty, -3) \cup (3, \infty)$

then $f'(x) = \left(\frac{1}{x^2-9}\right)(2x) = 0$

$2x = 0$

as $x^2 \neq 9$

$x = 0$ or for $x = 0$ $f'(x) = 0$

now $f'(x) = \dfrac{2x}{x^2 - 9}$ $\quad$ as $x \in (-\infty, -3) \cup (3, \infty)$

$x^2 \in (9, \infty)$

so denomintor always > 0

so for $x<0$ $f'(x)<0$
and $x>0$ $f'(x)>0$

$\therefore$ dec on $(-\infty, -3)$
and
inc on $(3, \infty)$

Now for local min/max: cannot have

(c) $f(x) = x^{2/3}(x-1)^4 \quad 0 \leq x \leq 1$

then $f'(x) = \frac{2}{3}x^{-1/3}(x-1)^4$
$+ 4x^{2/3}(x-1)^3$
$= 0$

$(x-1)^3\left[\frac{2}{3}\frac{(x-1)}{x^{1/3}} + 4x^{2/3}\right] = 0$

if $x \neq 0$

$(x-1)^3\left(\frac{2}{3}(x-1) + 4x\right) = 0$

so $f'(x) = 0$ for $x = 1$

or $2x - 2 + 12x = 0$
$x = 1/7$

Now as $0 \leq x \leq 1$, we will only see at $(1/7)$

now $f'(x) = (x-1)^3\left(\frac{2}{3}\frac{(x-1)}{x^{1/3}} + 4x^{2/3}\right)$

inc ≥ 0 for $x \in [0, 1/7]$

dec ≤ 0 for $x \in [1/7, 1]$

$f(1/7)$, $f(0)$, $f(1)$

$f(1/7)$ → local maximum

$f(0)$ → local minima

$f(1)$ → local minima

(d) $f(x) = (\sin x)/x$ if $x \neq 0$

$f(0) = 1 \qquad 0 \leq x \leq \pi/2$

$f'(x) = \frac{\cos x}{x} - \frac{\sin x}{x^2}$ for $x \neq 0$

$= \frac{x\cos x - \sin x}{x^2}$

as $x^2 > 0$

$x\cos x < \sin x$

for $x \in (0, \pi/2]$

$\Rightarrow f'(x)$ for $[0, \pi/2]$ ≤ 0

for $f'(0)$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{\frac{\sin x}{x}-1}{x}$$

$$= \lim_{x\to 0^+} \frac{\sin x - x}{x^2}$$

$$\left(\because \text{l'Hospital rule}\right) = \lim_{x\to 0^+} \frac{\cos x - 1}{2x}$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{2} = 0$$

so $f'(x) \le 0$ for $x \in [0, \pi/2]$

$\therefore$ dec, local max at 0, min at $\frac{\pi}{2}$

5.4 $f(x) = e^{-1/x^2}$ $x \neq 0$
$f(0) = 0$

a) f is cont $\forall x$:

as x is cont

$-1/x^2$ is cont for $\forall x \in \mathbb{R}\setminus\{0\}$

and e^x is cont $\forall x \in \mathbb{R}\setminus\{0\}$

so $f(x) = g\circ\left(\frac{-1}{x^2}\right) = e^{\frac{-1}{x^2}}$ is cont $\forall x \in \mathbb{R}\setminus\{0\}$

for $f(0) = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\frac{-1}{x^2}} = 0$$

so $f(0) = \lim_{x\to 0} f(x)$

$\therefore$ cont at $x = 0$

$\therefore$ cont at all $x \in \mathbb{R}$

(b) $f^{(n)}$ is cont for all x and $f^{(n)}(0) = 0$:

as $f(x)$ is cont

$f'(x)$ exist and

$x \neq 0$:

$$f'(x) = \frac{d}{dx}\left(e^{-1/x^2}\right)$$

$$= \frac{2}{x^3} e^{-1/x^2}$$

for $x = 0$

$$f'(0) = \lim_{x\to 0} \frac{f(x)}{x}$$

$$= \lim_{x\to 0} \frac{e^{-1/x^2}}{x}$$

Note $f(x) = \dfrac{1}{e^{\frac{1}{x^2}}} \leqslant \dfrac{1}{\frac{1}{n!}\left(\frac{1}{x^2}\right)^n} = n!\,(x)^{2n}$

as $e^{\frac{1}{x^2}} \geqslant \frac{1}{n!}\left(\frac{1}{x^2}\right)^n$ (taylor series)

$e^{\frac{-1}{x^2}} \leqslant n!\,x^{2n} \quad \forall n$

now $|f'(0)| = \left|\lim_{x\to 0} \frac{f(x)}{x}\right| \leqslant \left|\lim_{x\to 0} \frac{x^2}{x}\right| = 0$

$\left|\lim_{x\to 0} \frac{f(x)}{x}\right| \leqslant 0$

so $f'(0)=0$

so $f'(x) = \frac{2}{x^3} e^{-1/x^2} \quad x \neq 0$

$f'(x) = 0$ for $x=0$

is cont as

$\lim_{x\to 0} f'(x) = \frac{2}{x^3} e^{\frac{-1}{x^2}} = 0$

(from $n!\,x^{2n}$)

so true for $n=1$, suppose true for $n=k$, then for $n=k+1$:

finding $f'(x) = P_3\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$

$f''(x) = P_6\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$

$\vdots$

polynomial of degree $3k$

$f^k(x) = P_{3k}\left(\frac{1}{x}\right) e^{\frac{-1}{x^2}}$

then $f^{k+1}(x) = P_{3k+1}\left(\frac{1}{x}\right) e^{\frac{-1}{x^2}} + P_{3k+3}\, e^{-\frac{1}{x^2}}$

$= (P_{3k+3})\left(\frac{1}{x}\right) e^{\frac{-1}{x^2}}$

also $f^{k+1}(0) = 0$

and

$\lim_{x\to 0} f^{k+1}(x) = \dfrac{P_{3k+3}\left(\frac{1}{x}\right) e^{\frac{-1}{x^2}}}{x}$

$\leqslant \frac{(3k+3)!}{x} (x)^{2(3k+3)}$

$\leqslant 0$

so $(f^{k+1}(0))' = 0$

5.14 $f' \to$ finite in $x \in (0,1]$

s.t $|f'(x)| < 1$

$a_n = f\left(\frac{1}{n}\right) \quad n = 1, 2, \ldots\ldots$

To prove: $\lim_{n \to \infty} a_n$ exist

proof: $n \geq m$

by mvt

$$\frac{|a_n - a_m|}{|\frac{1}{n} - \frac{1}{m}|} \leq 1$$

$$\Rightarrow \left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right)\right| \leq \left|\frac{1}{n} - \frac{1}{m}\right|$$

then $\{a_n\}$ is cauchy as

$\left\{\frac{1}{n}\right\}$ is cauchy

so $\{a_n\}$ converges

or $\lim_{n \to \infty} a_n$ exist

5.15 f has finite derivative at each point on (a, b)

or use L' Hospital's rule
or darboux theorem

$\lim_{x \to c} f'(x)$ exist and finite for some $c \in (a, b)$

To prove: $\lim_{x \to c} f'(x) = f'(c)$

proof: now let $\lim_{x \to c} f'(x) = A$

then for $x \neq c$

$$\frac{f(x) - f(c)}{x - c} = f'(k)$$

for some $k \in (x, c)$ or (c, x)

by MVT

as $\lim_{x \to c} f'(x)$ exist

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$|x - c| < \delta \Rightarrow |f'(x) - A| < \varepsilon$

i.e for $x \in (c - \delta, c + \delta)$

$A - \varepsilon < f'(x) < A + \varepsilon$

so for $x \in (c-\delta, c+\delta)$ we have

$$\frac{f(x)-f(c)}{x-c} = f'(K)$$

$$A-\varepsilon < \frac{f(x)-f(c)}{x-c} < A+\varepsilon$$

or we have

$$A \leq f'(c) \leq A$$

$$\Rightarrow f'(c) = A = \lim_{x \to c} f'(x)$$

5.18 f has finite derivative in (a,b)
cont on $[a,b]$

s.t $f(a)=f(b)=0$

To prove: $\forall \lambda \in \mathbb{R}$, $\exists$ c in (a,b) s.t $f'(c) = \lambda f(c)$

proof: as $f'(c) = \lambda f(c)$

$$\Rightarrow \frac{f'(c)}{f(c)} = \lambda$$

$$\Rightarrow \ln|f(c)| = \lambda x$$

or $f(c) = e^{\lambda x}$ some form of this

or if $g(x) = f(x)e^{-\lambda x}$
then

$$g(a) = 0$$
$$g(b) = 0$$

so rolles theorm can be applied
and also

$$g'(x) = f'(x)e^{-\lambda x} - f(x)\lambda e^{-\lambda x} = 0$$

$$g'(c) = 0$$
some $c \in (a,b)$

$$\Rightarrow f'(c) = \lambda f(c)$$

5.20 f has finite third derivative in $[a,b]$ and if

$$f(a) = f'(a) = f(b) = f'(b) = 0$$

To show: $f'''(c) = 0$ for some $c \in (a,b)$

as $f(a) = f(b) \Rightarrow \exists$ $c_1 \in (a,b)$ s.t
$f'(c_1) = 0$

also as $f'(c_1)=0$

$\exists\, c_2 \in (a, c_1)$ s.t $f''(c_2)=0$

also $\exists\, c_3 \in (c_1, b)$ s.t $f''(c_3)=0$

so, $\exists\, c \in (c_2, c_3)$ s.t $f'''(c)=0$

Rolle's theorm used

5.21 f is non-negative ($f \geqslant 0$), and has finite third derivative f''' in open interval $(0,1)$.

if $f(x)=0$ for x_1, x_2 ($x_1 \neq x_2$) in $(0,1)$

as ($f \geqslant 0$) non-negative and and for $x_1, x_2 \in (0,1)$

then $f(x_1)=0$ and $f(x_2)=0$

or $\exists\, x_3 \in (x_1, x_2)$ s.t $f'(x_3)=0$ —— ①

now as local minimas

$f'(x_1)=0$ and $f'(x_2)=0$ —— ②

then $f''(x_4)=0$ and $f''(x_5)=0$

for $x_4 \in (x_1, x_3)$

$x_5 \in (x_3, x_2)$

and so $\exists\, c \in (x_4, x_5)$ s.t

$f'''(c)=0$ for some $c \in (0,1)$

Rolle's theorm used

Tutorial 9 :

6.1 B.V is $\sum |\Delta f_k| \le M$

monotonic $\Rightarrow$ BV
derivative is bounded $\Rightarrow$ B.V

(a) $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$$f'(x) = \begin{cases} 2x\sin(\frac{1}{x}) + \cancel{x^2}\cos(\frac{1}{x})(\frac{-1}{\cancel{x^2}}) & : x \neq 0 \\ x\sin(\frac{1}{x}) = 0 & ; x = 0 \end{cases}$$

$\le 2 + 1 = 3$

$|f'(x)| \le 3 \Rightarrow$ as f' is bounded it is B.V

(b) $f(x) = \begin{cases} \sqrt{x}\sin(\frac{1}{x}) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

nee Not B.V

as $P = \{0, \frac{2}{n\pi}, \frac{2}{(n-1)\pi}, \frac{2}{(n-2)\pi}, \dots \frac{2}{\pi}, 1\}$

$\quad\;\; = x_0, x_1, x_2, \dots\dots x_n, x_{n+1}$

$$|f_k - f_{k-1}| = \Delta f_k$$
$$= \left| f\left(\frac{2}{(n+1-k)\pi}\right) - f\left(\frac{2}{(n+1-k+1} \times \frac{1}{\pi}\right)\right|$$

$$|\Delta f_k| \ge \sqrt{\frac{2}{(n+2-k)} \cdot \frac{1}{\pi}}$$

as p series for $\frac{1}{n^p}$ $\quad p = \frac{1}{2}$
as $p < 1$

diverget

or

$$\underbrace{\sum \frac{c}{n^p}}_{\to \infty} < |\Delta f_k| \longrightarrow \infty$$

6.2 (a) $|f(x) - f(y)| < M\,|x-y|^\alpha$ on $\forall\, x, y \in [a,b]$

if $\alpha > 1$

then $\left|\frac{f(x)-f(y)}{x-y}\right| < M|x-y|^{\alpha-1}$

for $y = x+h$

$$\lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| < \lim_{h\to 0} M|h|^{\alpha-1}$$

$$\Rightarrow \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| = 0$$

$\Rightarrow f'(x) = 0 \;\forall\; x \in [a,b]$

$\Rightarrow f$ is cont

if $\alpha = 1$ then

as $|f'(x)| < M(h)^0 = M$

$\Rightarrow |f'(x)| < M$
$\Rightarrow f'$ is bounded
$\Rightarrow f$ is <u>B.V</u>

or $|\Delta f_k| < M\,|\Delta x_k|$

$\Sigma|\Delta f_k| < M\,\Sigma|\Delta x_k|$

$\Sigma|\Delta f_k| < M(b-a)$

6.11 $\forall\, \varepsilon > 0, \exists\, \delta > 0$ s.t

$\forall$ disjoint $\{(x_i, x_{i+1})\}$ sets

$\Sigma|\Delta x_k| < \delta \Rightarrow \Sigma|\Delta f_k| < \varepsilon$

<u>To prove</u>: Ab cont $\Rightarrow$ cont

<u>proof</u>:
if we take trivial partition $[a,b]$

$x_0 = a$
$x_1 = x$
$x_2 = p$
$x_3 = b$

then $|x-p| < \delta \Rightarrow |f(x)-f(p)| < \varepsilon$

<u>To prove</u>: Bounded variation if Ab. cont

<u>proof</u>: for BV $\forall\, P \in P[a,b]$

$\Sigma|\Delta f_k| < M$

now if $\varepsilon = 1$ then

$\Sigma |\Delta f_k| < 1$

$\exists \delta > 0$ s.t

$\Sigma |\Delta x_k| < \delta$

if $K \frac{\delta}{2} > (b-a)$

$\exists K$ s.t $K\left(\frac{\delta}{2}\right) > (b-a)$

Archimedean property

$x_0 = a$

$x_1 = a + \delta/2$

$\vdots$

$x_{k-1} = a + (k-1)\delta/2$

$x_k = a + (b-a) = b$

for $[x_i, x_{i+1}]$

any partition in $[x_i, x_{i+1}]$

$\Sigma |\Delta x| < \delta \Rightarrow \Sigma |\Delta f_k| < 1$

so sum of all $< \underbrace{1 + 1 + \cdots 1}_{k}$

$< k$

6.13 $\left.\begin{array}{l} f \text{ is ac} \\ g \text{ is ac} \end{array}\right\} [a,b]$
then

<u>to prove</u>: $|f|$ is a.c

<u>proof</u>: $\forall \{(x_i, x_{i+1})\} \rightarrow$ disjoint

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$\sum_i |\Delta x_i| < \delta$

$\Rightarrow \sum_i |\Delta f_i| < \varepsilon$

now

$|\Delta f_i| = |f_i - f_{i-1}|$

$\geq ||f_i| - |f_{i-1}||$

so $\sum_i |\Delta |f_i|| < \sum_i |\Delta f_i| < \varepsilon$

$\Rightarrow |f|$ is a.c

To prove: cf is A.C

proof: as f is A.C

$\forall \varepsilon > 0, \exists \delta > 0$ s.t

$$\sum |\Delta x_i| < \delta \Rightarrow \sum |\Delta f_i| < \varepsilon$$

for $\varepsilon' = \varepsilon / |c|$
we have

$\exists \delta > 0$ s.t

$$\sum |\Delta x_i| < \delta \Rightarrow \sum |\Delta f_i| < \varepsilon / |c|$$

$$\Rightarrow \sum |\Delta c f_i| < \varepsilon$$

To prove: $f+g$ is A.C

proof: f is AC, g is AC

$\forall \varepsilon > 0, \exists \delta > 0$ s.t
$\forall \varepsilon' > 0, \exists \delta' > 0$ s.t

$$\sum |\Delta x_i| < \delta \Rightarrow \sum |\Delta f_i| < \varepsilon$$

and

$$\sum |\Delta x_i| < \delta' \Rightarrow \sum |\Delta g_i| < \varepsilon'$$

now for $\{(x_i, x_{i+1})\} \leftarrow$ disjoint sets

let $\delta = \min\{\delta, \delta'\}$

then

$$\sum |\Delta x_i| < \delta \Rightarrow \sum |\Delta f_i| < \varepsilon \ \& \ \sum |\Delta g_i| < \varepsilon'$$

$$\sum |\Delta f_i + \Delta g_i| \leq \sum |\Delta f_i| + |\Delta g_i| < \varepsilon + \varepsilon'$$

then for $\underset{=\varepsilon}{\varepsilon/2}$ and $\varepsilon/2 = \varepsilon'$

we get $\sum |\Delta f_i + \Delta g_i| < \varepsilon$

To prove: $f.g$ is AC

proof:

$$\sum |f g_i - f g_{i-1}|$$

$$= \sum | g_i (f_i - f_{i-1}) + f_{i-1}(g_i - g_{i-1}) |$$

$$\leq \sum |g_i| \max |f_i - f_{i-1}| + \sum |f_{i-1}| \max |g_i - g_{i-1}|$$

$$\leq \sum M_g |f_i - f_{i-1}| + \sum M_f |g_i - g_{i-1}|$$

$M_g = \sup |g(x)| \quad \forall x \in [a,b]$
as g is bounded
(A.C)

$$\leq M_g\left(\frac{\varepsilon}{2(M_g+1)}\right) + \frac{M_f\,\varepsilon}{(2(M_f+1))}$$

$$< \varepsilon$$

To prove: f/g is B.V

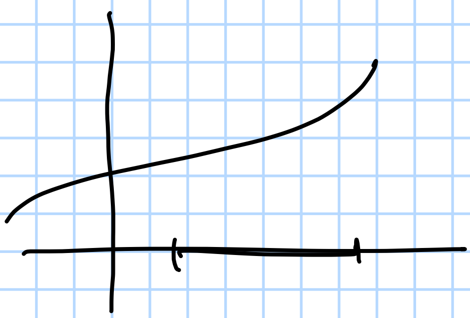
proof:

as g is s.t
$0 < m \leq g(x)$
then
$$\frac{1}{g(x)} \leq \frac{1}{m}$$

then $\left| 1/g_i - 1/g_{i-1} \right|$
$$\leq \frac{1}{m^2} |\Delta g_i|$$

so $1/g$ is B.V
then
$f \times 1/g$ is B.V

4.62 f is one-one and cont on $[a,b]$
then $\Rightarrow$ monotonic



f is cont on $[a,b]$
we have
max and min exist

let $M = \sup\{f(x) \mid \forall x \in [a,b]\}$
$m = \inf\{f(x) \mid \forall x \in [a,b]\}$

$M(f) = f(p) \quad \exists p \in [a,b]$
$m(f) = f(q) \quad \exists q \in [a,b]$

assume $p \in (a,b)$ then
$\exists \delta > 0$ s.t
$f(y) \leq f(p) \quad \forall \quad y \in (p-\delta, p+\delta) \subseteq [a,b]$

$y_1 \in (x-\delta, x)$
& $y_2 \in (x, x+\delta)$

as $1-1 \Rightarrow f(y_1) < f(x)$
& $f(y_2) < f(x)$

$f(y_1) < \gamma < f(x) \Rightarrow \gamma = f(z_1) \quad z_1 \in (y_1, x)$
$f(y_2) < \gamma < f(x) \Rightarrow \gamma = f(z_2) \quad z_2 \in (x, y_2)$

this is a contradiction of 1-1

so $p \notin (a,b)$
or $p \in \{a,b\}$
similarly $q \in \{a,b\}$

wlog f is strictly dec on $[a,b]$
when $p = \{a\}$
$q = \{b\}$

then <u>suppose not</u> :

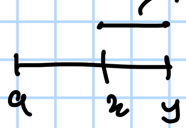
$\exists x, y \in [a,b]$
with $x < y$ s.t $f(x) < f(y)$
(= does not hold as 1-1)

now $[x,y]$ $f|_{[x,y]}$ has max at y
& $f|_{[a,y]}$ has min at y

or $B(y,\delta) \cap [x,y]$ s.t f is <u>const</u> on $B(y,\delta) \cap [x,y]$

which is a contradiction to f being 1-1

$\Rightarrow$



contradiction

$\therefore f$ is monotonically dec

4.65 f is strictly inc on subset S of $\mathbb{R}$

(a) $f(S)$ is open $\Rightarrow f$ is conti ous

<u>To prove</u>: $\forall \varepsilon > 0, \exists \delta > 0$ s.t
$x \in B(a,\delta) \cap S \Rightarrow f(x) \in B(f(a), \varepsilon)$

as $f(S)$ is open, $\exists r > 0$ s.t
$B(f(a), r) \subseteq f(S)$

<u>case I</u> : for $\varepsilon > r$; $B(f(a), r) \subseteq B(f(a), \varepsilon)$

Here $f(B(a, \delta) \cap S) \subseteq B(f(a), r)$ then we are done

$y_1 = f(a) - r/2$

$y_2 = f(a) + r/2$

& $y_1 = f(x_1)$

$y_2 = f(x_2)$

doubt

we have $x_1 < a < x_2$ (as monotonic inc)

$x \in (x_1, x_2) \cap S$

s.t $f(x_1) < f(x) < f(x_2)$

$\forall f(x) \in B(f(a), r)$

$\Rightarrow f(x_1) < f(x) < f(x_2)$

$\Rightarrow \delta = \min(a - x_1, x_2 - a)$

then

$B(a, \delta) \cap S = (a-\delta, a+\delta) \cap S$

$\subseteq (x_1, x_2) \cap S$

now $B(a, \delta) \cap S \subseteq (x_1, x_2) \cap S$

we have

$f(B(a, \delta) \cap S) \subseteq f((x_1, x_2) \cap S)$

$= B(f(a), r)$

$\Rightarrow f(B(a, \delta) \cap S) \subseteq B(f(a), r) \subseteq B(f(a), \varepsilon)$

$\therefore$ f is cont

<u>case II</u> : $r > \varepsilon$

in this case as $r > \varepsilon$ we have

r s.t

$B(f(a), r) \subseteq f(S)$

then as $B(f(a), r) \subseteq f(S)$

$\Rightarrow B(f(a), \varepsilon) \subseteq f(S)$

or new $r = \varepsilon/2$ (any other value)

$\therefore$ again $r < \varepsilon$ so same result doubt

(b) $f(S)$ is connected $\Rightarrow$ f is cont

as $f(S)$ is connected
there is some interval
say I

now $a \in S$ s.t
$F(a) \in I$

a can be ① $f(a)$ is interior point
— same as (a)

② $f(a)$ is end point
— same as (a)

Tutorial-10 :

1. $x_0 \in [0,1]$ $\quad f:[0,1] \to \mathbb{R}$
$$f(x_0)=1$$
$$\text{\&} \quad f(x)=0 \quad x \neq x_0$$

f is Riemann integrable as

$$P_\varepsilon = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \ldots \tfrac{i}{n}, x_0, \tfrac{i+1}{n}, \ldots 1\}$$

then $\quad L(P_\varepsilon, f) = 0$

$$U(P_\varepsilon, f) = \frac{1}{n}$$

as $n \to \infty$

$U(P_\varepsilon, f) \to L(P_\varepsilon, f) \Rightarrow f$ is riemann integrable

$$\therefore U(f) = L(f) = \int_0^1 f(x)\,dx = 0$$

2. $f:[a,b] \to \mathbb{R}$ cont. function
$f(x) \geq 0 \quad \forall x \in [a,b]$

To show: $\int_a^b f(x)\,dx = 0 \Rightarrow f = 0$

if $f \neq 0$ at say c then

$$f(c) \neq 0$$

then as cont then
$\forall \varepsilon > 0, \exists \delta > 0$ s.t
$\forall |x-c| < \delta$
$|f(x) - f(c)| < \varepsilon$
for $\varepsilon = |f(c)|/2$
we get

$$|f(c)| - |f(x)| < |f(x) - f(c)| < \frac{|f(c)|}{2}$$

$$\Rightarrow \frac{|f(c)|}{2} < |f(x)|$$

$$\Rightarrow |f(x)| > \frac{|f(c)|}{2}$$

so $f(x) > \frac{|f(c)|}{2}$ or $f(x) < -\frac{|f(c)|}{2}$

for $f(x) > \frac{|f(c)|}{2}$ we get this some $x \in B(c, \delta)$

$$U(P, f) > (\delta)\left(\frac{|f(c)|}{2}\right) \quad \forall P$$

$$\Rightarrow U(f) > \delta \frac{|f(c)|}{2} > 0$$

$$\Rightarrow \int_a^b f(x)\,dx > 0 \quad \text{(contradiction)}$$

Same for $\quad f(x) < -\frac{|f(c)|}{2}$

3. $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$

using FTC on $\int_0^x e^{t^2}dt$ (as e^{t^2} is cont)

we get $F(x)=\int_0^x e^{t^2}dt$

$\Rightarrow F'(x)=e^{x^2}$

as $x\to 0$ $F(x)\xrightarrow[x\to 0]{} 0$

& $x \xrightarrow[x\to 0]{} 0$

using l'Hospital rule

we get

$$\lim_{x\to 0}\frac{1}{x}\int_0^x e^{t^2}dt = \lim_{x\to 0} e^{x^2} = 1$$

4. $f'(x)$ if $f(x)=\int_0^{x^3}\frac{1}{1+\sin^2 t}\,dt$

$g(x)=\int_0^x \frac{1}{1+\sin^2 t}$

then we can use FTC as $\frac{1}{1+\sin^2 t}$ is cont

to get $g'(x)=\frac{1}{1+\sin^2 x}$

we have to find $(g(x^3))' = g'(x^3)[3x^2]$ (∵ chain rule)

$=(f(x))' = \frac{1}{1+\sin^2 x^3}(3x^2)$

5. $\frac{8}{15\sqrt5} \le \int_0^{\pi/2}\frac{\cos^5 x}{\sqrt{1+x^2}}\,dx \le \frac{8}{15}$

"$f(x)$"

$f'(x)\le 0 \ \forall \ x\in[0,\pi/2)$

① $f(x)\ge 0 \ \forall \ x\in[0,\pi/2]$

② $f'(x)\le 0 \ \forall \ x\in[0,\pi/2]$

now $\frac{\cos^5 x}{\sqrt{1+x^2}} \le \cos^5 x$

$\Rightarrow \int_0^{\pi/2}\frac{\cos^5 x}{\sqrt{1+x^2}} \le \int_0^{\pi/2}\cos^5 x\,dx$

where $\int_0^{\pi/2}\cos^5 x\,dx = \int_0^{\pi/2}(1-\sin^2 x)^2\cos x\,dx$

$=\int_0^1 (1-t^2)^2 dt = \int_0^1 1+t^4-2t^2\,dt = 1+\frac15-\frac25 = \frac{8}{15}$

or $\int_0^{\pi/2} \frac{\cos^5 x}{\sqrt{1+x^2}} \leq 8/15$

now as $\sqrt{1+x^2}$ is inc
max value is at $\pi/2$

$$\sqrt{1+(\pi/2)^2}$$

or $\sqrt{1+x^2} \leq \sqrt{1+(\pi/2)^2} \leq \sqrt{1+(2)^2} = \sqrt{5}$

$$\Rightarrow \frac{1}{\sqrt{5}} \leq \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{\cos^5 x}{\sqrt{5}} \leq \frac{\cos^5 x}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{8}{15\sqrt{5}} \leq \int_0^{\pi/2} \frac{\cos^5 x}{\sqrt{1+x^2}} \leq \frac{8}{15}$$

6. f is rieman int on $(0,1]$
or on $[c,1]$ for $c>0$

$$\int_0^1 f(x)dx = \lim_{c\to 0} \int_c^1 f(x)dx$$

(a) Given f is Rieman int on $[0,1]$ we get
$\exists P_\varepsilon \in P[0,1]$
s.t
$|L(P_\varepsilon, f) - U(P_\varepsilon, f)| < \varepsilon/2$

now
let $P_\varepsilon' \in P[c,1]$
then
$|U(P_\varepsilon', f) - L(P_\varepsilon', f)| < \varepsilon/2$

now if $P = \{0,c\} \cup P_\varepsilon'$
then

$$U(\{0,c\}, f) + U(P_\varepsilon', f) = U(P,f) \geq U(P_\varepsilon, f)$$
$$L(\{0,c\}, f) + L(P_\varepsilon', f) = L(P,f) \leq L(P_\varepsilon, f)$$

$$U(P_\varepsilon', f) \geq U(P_\varepsilon, f) - U(\{0,c\}, f)$$
$$-L(P_\varepsilon', f) \geq -L(P_\varepsilon, f) + L(\{0,c\}, f)$$

$$\frac{\varepsilon}{2} > |U(P_\varepsilon, f) - L(P_\varepsilon, f)| + |U(\{0,c\}, f) - L(\{0,c\}, f)|$$

$$\Rightarrow |U(\{0,c\}, f) - L(\{0,c\}, f)| < \varepsilon/2$$

$\Rightarrow \int_0^c f(x)dx$ exist and is equal to zero from defination

$\therefore$ agrees with old one

(b) $\int_0^1 f(x)dx = \lim_{c\to 0} \int_c^1 f(x)dx$ $\quad$ $f \to$ above limit exist
$|f| \to$ does not

$f(x) = \begin{cases} \frac{2^n}{n}(-1)^{n+1} ; x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \end{cases}$

$$\int_{1/2^n}^{1/2^{n-1}} f(x)\,dx = \left[\frac{1}{2^{n-1}} - \frac{1}{2^n}\right]\left[\frac{2^n}{n}(-1)^{n+1}\right]$$

$$= \left[\frac{1}{\cancel{2^n}}\right]\left[\frac{\cancel{2^n}}{n}(-1)^{n+1}\right]$$

$$= \frac{(-1)^{n+1}}{n}$$

$$\lim_{c \to 0} \int_c^1 f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \to \text{converges}$$

$$\text{but} \lim_{c \to 0} \int_c^1 |f(x)|\,dx = \sum_{n=1}^{\infty} \frac{1}{n} \to \text{diverges}$$

7. (a) $\int_{-\infty}^{\infty} e^{-x^2} dx$ now

e^{-x^2} is a dec function

s.t

as $x \to \infty$ $e^{-x^2} \to 0$

then

$\exists x_0 \in \mathbb{R}$ s.t

$|e^{-x^2} - 0| < \varepsilon$

$\forall x \geq x_0$

for $\varepsilon' = \frac{\varepsilon}{|x-y|}$

then $\int_x^y e^{-x^2} dx \leq \int_x^y \frac{\varepsilon}{x-y} dx = \varepsilon$

or $\forall x, y \geq x_0$, $\exists x_0$ s.t

$$\left|\int_x^y e^{-x^2} dx\right| \leq \varepsilon$$

$\Rightarrow$ cog for $\int_0^{\infty} e^{-x^2} dx$

same for $\int_{-\infty}^{0} e^{-x^2} dx$

or $\int_{-\infty}^{\infty} e^{-x^2} dx \to$ cong

(b) $\int_1^b \frac{dx}{\log(x)}$ $b>1$

is convg if

$\lim_{c\to 1^+} \int_c^b \frac{dx}{\log(x)}$ exist

$f(x) = \frac{1}{\log(x)}$ $g(x) = \frac{1}{x}\frac{1}{\log(x)}$

then

$\int_1^b \frac{1}{x}\frac{1}{\log(x)} dx = \int_0^{\log(b)} \frac{dt}{t} = \left|\log(t)\right|_0^{\log(b)}$

$= \log(\log(b)) - \log(0)$ $\rightarrow$ divg

$g(x) = \frac{1}{x}\frac{1}{\log x}$ divg

now $\lim_{x\to 1^-} \left|\frac{f(x)}{g(x)}\right| = |x| = 1$

so by limit comparison $\int_1^b \frac{1}{\log x} dx$

also diverges

(c) $\int_0^1 \frac{dx}{\log(x)}$

here $g(x) = \frac{1}{(x+1)\log x}$

$\int_0^1 \frac{1}{(x+1)}\frac{1}{\log x} dx$

$\log(x+1) = t$

$\frac{1}{x+1} dx = dt$ $x+1 = e^t$

$x = e^t - 1$

$\int_0^1 \frac{dt}{\log(e^t-1)}$

$= \int_1^2 \frac{dt}{\log(e^t-1)}$

as convg for $\int_1^2 \frac{dt}{\log(e^t-1)}$

we have $g(x)$ as convg

then by limit comparison test $f(x)$ is also convg

8. (a) let $\int_0^t e^{-x} x^2 dx$

$$= x^2 \int_0^t e^{-x} dx - \int_0^t 2x \frac{e^{-x}}{-1} dx$$

$$= \left| x^2 \frac{e^{-x}}{-1} \right|_0^t + \int_0^t 2x\, e^{-x} dx$$

$$= -t^2 e^{-t} + 2\left[\left| x\frac{e^{-x}}{-1} \right|_0^t + \int_0^t \frac{e^{-x}}{+1} dx \right]$$

$$= -t^2 e^{-t} + 2\left[-te^{-t} + \left| \frac{e^{-x}}{-1} \right|_0^t \right]$$

$$= -t^2 e^{-t} + 2\left[-te^{-t} - e^{-t} + e^0 \right]$$

$$= \underbrace{-t^2 e^{-t}}_{\to 0} \underbrace{-2te^{-t}}_{\to 0} \underbrace{-2e^{-t}}_{\to 0} + 1$$

as $t \to \infty$

or for $\lim_{t \to \infty} \int_0^t x^2 e^{-x} dx = 1$

(b) $$\int_0^4 \frac{dx}{\sqrt{x(4-x)}} = \int_0^4 \frac{dx}{\sqrt{4x - x^2}} = \int_0^4 \frac{dx}{\sqrt{4-(x-2)^2}} = \int_0^4 \frac{dx}{\sqrt{2^2-(x-2)^2}}$$

$x^2 - 4x + 4 - 4$
$= (x-2)^2 - (2)^2$

$x - 2 = u$
$dx = du$

$$\int_{-2}^{2} \frac{du}{\sqrt{4-u^2}} = \int_{-2}^{2} \frac{du}{2\sqrt{1-\left(\frac{u}{2}\right)^2}}$$

$\frac{u}{2} = t$

$$= \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(1) - \sin^{-1}(-1)$$

$= \pi/2 - (-\pi/2)$
$= \pi$

(c) $\int_1^\infty \frac{dx}{x \log(x)}$

for $$\int_a^b \frac{dx}{x\log(x)} = \int_{\log a}^{\log b} \frac{dt}{t} = \log(|\log(b)|) - \log(|\log(a)|)$$

for $a = 1$
$b = \infty$

we have $\to \infty + \infty \to \infty$
or div for 1 to ∞

9. $$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2} = \sum \frac{1}{n} \times \frac{1}{1+\left(\frac{i}{n}\right)^2}$$

$$= \int_0^1 \frac{1}{1+x^2} \, dx$$

$$= \left. \tan^{-1} x \right|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \pi/4 - 0 = \pi/4$$

Tutorial-11 :

1. $f : [-\pi, \pi] \longrightarrow \mathbb{R}$
$f(x) = \sin(x)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(1.x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(1.x)\cos(n.x)\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(1.x)\sin(n.x)\,dx = \begin{cases} 1; & n=1 \\ 0; & n\neq 1 \end{cases}$$

$$\therefore f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$$
$$= 0 + 0 + 1.\sin(x)$$
$$= \sin x$$

<u>method 2</u>: $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\sin t]\, e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\sin t][\cancel{\cos nt} - i \sin nt]\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [-i]\sin nt.\sin t\,dt$$

$n \neq 1, -1$ $\qquad$ $n = 1, -1$

$\alpha_n = 0$ $\qquad$ $\alpha_1 = \frac{1}{2\cancel{\pi}}(-i)(\cancel{\pi}) = \frac{-i}{2}$

$\alpha_{-1} = i/2$

or $f(x) = \alpha_1 e^{i(1)x} + \alpha_{-1} e^{i(-1)x} = \frac{-i}{2} e^{ix} + \frac{i}{2} e^{-ix}$

$$= i/2\,[-e^{ix} + e^{-ix}]$$

$$= \frac{1}{2i}\left[e^{ix} - e^{-ix}\right]$$

$$f(x) = \sin x$$

9.1: $f_n \to f$ uniformly on S
each f_n is bounded on S

$\forall n \in \mathbb{N}, \exists \alpha_n$ s.t
$|f_n(x)| \leq \alpha_n \ \forall x \in \mathbb{R}$

To prove: $\{f_n\}$ is uniformly bounded on S.

$\left(|f_n(x)| < M \ \forall n = 1, 2 \dots \right)$

proof: as $f_n \longrightarrow f$
$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t
$\forall n \geq n_0$
$|f_n - f| \leq \varepsilon \ \forall x \in \mathbb{R}$

for $\varepsilon = 1 \Rightarrow |f_n - f| \leq 1$
$\Rightarrow |f_n| - |f| \leq 1$
$\Rightarrow |f_n| \leq 1 + |f|$

$|f| \leq 1 + |f_{n_0}| \leq \alpha_{n_0} + 1$
or
f is also bounded
now $|f_n| \leq 1 + |f| \leq \alpha_{n_0} + 2$
or
$\exists n_0 \in \mathbb{N}$ s.t
$|f_n| \leq 1 + |f| \ \forall n \geq n_0$

let $M = \max\{\alpha_1, \alpha_2, \dots \alpha_{n_0 - 1}, \underset{\underset{2 + \alpha_{n_0}}{\parallel}}{1 + |f|}\}$
then
$|f_n| \leq M \ \forall n \in \mathbb{N}$

$M = \max\{\alpha_1, \alpha_2, \dots \alpha_{n_0 - 1}, \alpha_{n_0} + 2\}$

9.2: $\{f_n\}$ and $\{g_n\}$
$f_n(x) = x\left(1 + \frac{1}{n}\right) \quad x \in \mathbb{R}, n = 1, 2, \dots$

$$g_n(x) = \begin{cases} \frac{1}{n} & ; x = 0 \text{ or } x \in \mathbb{R} \setminus \mathbb{Q} \\ b + \frac{1}{n} & ; x \in \mathbb{Q} \text{ say } x = \frac{a}{b}, b > 0 \end{cases}$$

$h_n(x) = f_n(x)\, g_n(x)$

(a) To prove: Both $\{f_n\}$ & $\{g_n\}$ converges uniformly on every bounded interval.

$f_n(x) = x\left(1 + \frac{1}{n}\right)$ for $n \to \infty$
$f_n(x) \longrightarrow x$ (pointwise)

$|f_n(x) - x| = \left|\frac{x}{n}\right|$ Bounded int
as $x \in [-M, M]$

$\left|\frac{x}{n}\right| \leq \frac{M}{n}$ By archimedean property
$M/n_0 < \varepsilon$ ($\exists n_0$)

$\therefore \forall n \geq n_0 \ |f_n(x) - x| < \varepsilon \ \forall n \geq n_0, \forall x \in [-M, M]$

now $g_n(x) \longrightarrow \begin{cases} 0 & ; x=0,\ x \in \mathbb{R}\setminus\mathbb{Q} \\ \frac{1}{b} & ; x \in \mathbb{Q},\ x = \frac{a}{b} \end{cases}$

$$\left| g_n(x) - g(x) \right| = \left| \frac{1}{n} \right| < \varepsilon \quad \text{(Archimedian property)}$$

$\therefore \{f_n\}, \{g_n\}$ conv uniformly on every bounded interval.

(b) <u>To prove</u>: $h_n(x)$ does not converge uniformly on any bounded interval.

$$h_n(x) = \begin{cases} \frac{x}{n}\left(1+\frac{1}{n}\right) ; & x=0 \text{ or } x \text{ is irrational} \\ \left(b+\frac{1}{n}\right)\left(x\right)\left(1+\frac{1}{n}\right) ; & x \in \mathbb{Q},\ x = a/b,\ b>0 \end{cases}$$

$$= \begin{cases} \frac{x}{n}\left(1+\frac{1}{n}\right) ; & x=0 \text{ or } x \text{ is irrational} \\ \left(b+\frac{1}{n}\right)\left(x\right)\left(1+\frac{1}{n}\right) ; & x \in \mathbb{Q},\ x = \frac{a}{b} \end{cases}$$

$$h_n(x) \longrightarrow \begin{cases} 0 ; & x=0 \text{ or } x \in \mathbb{R}\setminus\mathbb{Q} \\ bx ; & x \in \mathbb{Q},\ x=\frac{a}{b},\ b>0 \end{cases} = h(x)$$

now $\left| h_n(x) - h(x) \right|$

$$= \begin{cases} \left|\frac{x}{n}\right| \left|1+\frac{1}{n}\right| ; & x=0 \text{ or } x \in \mathbb{R}\setminus\mathbb{Q} \\ \left|\left(bx+\frac{b}{n}\right)\left(1+\frac{1}{n}\right) - bx\right| ; & x \in \mathbb{Q},\ x=\frac{a}{b},\ b>0 \end{cases}$$

$$= \begin{cases} \left|\frac{x}{n}\right| \left|1+\frac{1}{n}\right| ; & x=0 \text{ or } x \in \mathbb{R}\setminus\mathbb{Q} \\ \left|\cancel{bx} + \frac{bx}{n} + \frac{b}{n} + \frac{b}{n^2} - \cancel{bx}\right| ; & x \in \mathbb{Q},\ x=\frac{a}{b},\ b>0 \end{cases}$$

$$= \begin{cases} \left|\frac{x}{n}\right| \left|1+\frac{1}{n}\right| ; & x=0 \text{ or } x \in \mathbb{R}\setminus\mathbb{Q} \\ \left|\frac{bx}{n} + \frac{b}{n} + \frac{b}{n^2}\right| ; & x \in \mathbb{Q},\ x=\frac{a}{b},\ b>0 \end{cases}$$

if $h_n(x)$ conv uniformly on $[c,d]$ then let

$$\varepsilon = \max(|c|, |d|) > 0$$

$\exists\, n_0 \in \mathbb{N}$ s.t

$$|h_n(x) - h(x)| < \varepsilon \quad \forall\, n \geq n_0$$

$\left|\frac{x}{n}\right| \leq \left|\frac{x}{n}\right| \left|1+\frac{1}{n}\right| < \varepsilon$ & $\left|\frac{bx}{n} + \frac{b}{n} + \frac{b}{n^2}\right| < \varepsilon$

$\Rightarrow \left|\frac{x}{n}\right| < \varepsilon \ \forall\, n \geq n_0$ $\quad |b| \left|\frac{x}{n} + \frac{1}{n} + \frac{1}{n^2}\right| < \varepsilon$

$\varepsilon = \max\{|c|, |d|\}$

as $\left|\frac{x}{n}\right| < \max\{|c|, |d|\}$

$$\left|\frac{x}{n}\right| |b| < |b| \left| \frac{x}{n} + \frac{1}{n} + \frac{1}{n^2} \right| < \max\{|c|, |d|\}$$

$$|b| \left|\frac{x}{n}\right| < \max\{|c|, |d|\}$$

for $b = n^2 \Rightarrow \quad |x| \leq 0 \Rightarrow |x| = 0$

$\left(|x| < \frac{\max\{|c|, |d|\}}{n} \right)$ which is not true

9.5 (a) $f_n(x) = \frac{1}{nx+1}$; $0 < x < 1$, $n = 1, 2 \ldots$

$\lim_{n\to\infty} f_n(x) = \frac{1}{nx+1} \quad x \in (0,1)$

$\rightarrow 0$

$\therefore$ $f_n(x)$ conv pointwise for $x \in (0,1)$

$f_n(x) \xrightarrow{\text{Ptwise}} 0$

lets suppose it conv uniformly on $x \in (0,1)$, then

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ s.t

$\forall n \geq n_0$

$|f_n(x) - f(x)| < \varepsilon \ \forall x \in (0,1)$

$\Rightarrow \left| \frac{1}{nx+1} - 0 \right| < \varepsilon$

$\Rightarrow \frac{1}{(nx+1)} < \varepsilon$

for $\varepsilon = \frac{1}{2}$ but $\frac{1}{n_0 x+1} < \frac{1}{2}$

for $x = 0^+$

$\Rightarrow \frac{1}{0+1} = 1 \nless \frac{1}{2}$

this is a contradiction

(b) $g_n(x) = \frac{x}{nx+1}$; $0 < x < 1$, $n = 1, 2 \ldots$

To prove : $g_n \longrightarrow 0$ uniformly on $(0,1)$

proof : $g_n \xrightarrow{\text{Pt wise}} 0$

now, $|g_n - 0| = \left| \frac{x}{nx+1} - 0 \right|$

$= \frac{x}{nx+1}$ as $n > 0$, $x > 0 \Rightarrow \frac{x}{nx+1} > 0$

now $\frac{x}{nx+1} < \left|\frac{1}{n}\right| < \varepsilon$ (archimedian priciple)

9.7 : $f_n \to f$ uniformly on S
f_n is cont on S $\forall n \in \mathbb{N}$

$x \in S$, let $\{x_n\}$ be seq of points in S s.t $x_n \to x$

To prove : $f_n(x_n) \longrightarrow f(x)$

proof : as f_n are cont, f is also cont & so
$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t for x

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$$

also as $f_n(x) \longrightarrow f(x)$
$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$

$$|f_n(x) - f(x)| < \varepsilon/2$$
$$\forall x \in S$$

also as $x_n \to x$
$\forall \varepsilon > 0$, $\exists n_1 \in \mathbb{N}$ s.t
$|x_n - x| < \delta$

now, $$\left|f_n(x_n) - f(x)\right| \leq |f_n(x_n) - f_n(x) + f_n(x) - f(x)|$$
$$\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$
for $n \geq \max\{n_0, n_1\}$

$\therefore |f_n(x_n) - f(x)| < \varepsilon$ for $\forall \varepsilon$
$\Rightarrow f_n(x_n) \longrightarrow f(x)$

9.14 : $f_n(x) = \dfrac{x}{1+nx^2}$ $\quad x \in \mathbb{R}$
$n = 1, 2, \ldots$

$\lim_{n\to\infty} f_n = 0$

$$f'_n = \frac{1+nx^2}{(1+nx^2)^2} - \frac{x(2nx)}{(1+nx^2)^2}$$

$$= \frac{1 - nx^2}{(1+nx^2)^2} \to \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

(a) $f'(0) = 0 \neq g(0)$

$f'(x) = g(x)$ $\forall x \neq 0$

(b) $f_n \to f$ uniformly

$$|f_n - f| = \left|\frac{x}{1+nx^2}\right|$$

Here $1+nx^2 \geq 2\sqrt{nx^2}$

$$\Rightarrow \frac{1+nx^2}{2} \geq \sqrt{n}\,|x|$$

$$\Rightarrow \frac{2}{1+nx^2} \leq \frac{1}{\sqrt{n}\,|x|}$$

so $\left|\frac{x}{1+nx^2}\right| \leq \frac{|x|}{2\sqrt{n}\,|x|} = \frac{1}{2\sqrt{n}}$

$\therefore f_n \to f$ uniformly on $\mathbb{R}$

(c) $f_n' \longrightarrow g$ uniformly

$g(x) = \begin{cases} 1 ; & x = 0 \\ 0 ; & x \neq 0 \end{cases}$

$$\left|\frac{1-nx^2}{(1+nx^2)^2} - g(x)\right|$$

for $x = 0 \Rightarrow \left|\frac{1}{1} - 1\right| = 0$ always $0 < \varepsilon$

$I = [a, b]$

for $x \neq 0$

$$\left|\frac{1-nx^2}{(1+nx^2)^2} - 0\right| \leq \left|\frac{\cancel{1+nx^2}}{(1+nx^2)^{\cancel{2}}}\right| = \left|\frac{1}{1+nx^2}\right| \leq \frac{1}{|nx^2|} \leq \frac{1}{na^2}$$

$f_n' \to g$ uniformly on $I = [a, b]$

which does not contain 0.

9.15 : $f_n(x) = \left(\frac{1}{n}\right) e^{-n^2x^2}$ $\forall\, x \in \mathbb{R}, n = 1, 2, \ldots$

To prove : $f_n \longrightarrow 0$ uniformly on $\mathbb{R}$

$f_n' \longrightarrow 0$ pointwise on $\mathbb{R}$

$\{f_n'\}$ conv not uniform on any interval cont 0.

proof : $f_n \longrightarrow 0$ pointwise

$$\left|\frac{1}{n} e^{-n^2x^2}\right| \leq \frac{1}{n} < \varepsilon$$

as $e^{-n^2x^2} \leq 1$

also $f_n' = \frac{1}{\cancel{n}} e^{-n^2x^2}[-n^{\cancel{2}}][2x]$

$= -2nx\, e^{-n^2x^2} \longrightarrow 0$ pointwise

if uniform conv on say I $(0 \in I)$ then:

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ s.t

$$|2n_0 x e^{-n_0^2x^2}| < \frac{1}{e} \quad \left(\text{for } \varepsilon = \frac{1}{e}\right)$$

$$\Rightarrow |2n_0 x e^{-n_0^2x^2}| < \frac{1}{e}$$

for $x = \frac{1}{n_0} \Rightarrow \left|2\cancel{n_0}\frac{1}{\cancel{n_0}} \cdot e^{-\cancel{n_0^2}\frac{1}{\cancel{n_0^2}}}\right| = \frac{2}{e} < \frac{1}{e}$

contradiction

9.16 : $\{f_n\}$ seq of real-valued cont. functions defined on $[0,1]$

$f_n \to f$ uniformly on $[0,1]$

To prove : $\lim_{n\to\infty} \int_0^{1-1/n} f_n(x) = \int_0^1 f(x)dx$

proof : as $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$

& $\int_{1-1/n}^{1} f_n(x)dx$ & f_n has upper bound

$|f_n(x)| \le M \quad \forall x \in [0,1]$

$$\left| \int_{1-1/n}^{1} f_n(x)\,dx \right| \le \frac{M}{n} \to 0$$

$$\therefore \lim_{n\to\infty} \int_0^{1-1/n} f_n(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx$$

$$= \int_0^1 f(x)dx$$

9.19 To prove : $\sum_{n=1}^{\infty} \frac{x}{n^\alpha} \times \frac{1}{(1+nx^2)}$ conv uniformly on every finite interval in $\mathbb{R}$ if $\alpha > 1/2$

proof : $$\left| \frac{x}{n^\alpha(1+nx^2)} \right| \le \frac{1}{2n^{\alpha+1/2}} \quad \forall x \in \mathbb{R}$$

(AP ≥ GP)

Hence By weierstrass M-test

$$\sum \frac{1}{2n^{\alpha+1/2}} < \infty$$

for $\alpha + 1/2 > 1$
$\Rightarrow \alpha > \frac{1}{2}$

9.22 To prove : $\sum a_n \sin nx$, $\sum b_n \cos nx$ are uniformly conv for $\sum |a_n|$ conv

proof : $|a_n \sin nx| \le |a_n|$
By weierstrass M-test

$|b_n \cos nx| \le |b_n|$
By weierstrass M-test